NAKAI-MOISHEZON CRITERION FOR ADELIC $\mathbb{R}$-CARTIER DIVISORS

FRANÇOIS BALLAŸ

Université Clermont Auvergne, CNRS, LMBP, F-63000 Clermont-Ferrand, France.
francois.ballay@uca.fr

Abstract. We prove a Nakai-Moishezon criterion for adelic $\mathbb{R}$-Cartier divisors, which is an arithmetic analogue of a theorem of Campana and Peternell. Our main result answers a question of Burgos Gil, Philippon, Moriwaki and Sombra. We deduce it from the case of adelic Cartier divisors (due to Zhang) by continuity arguments and reductions involving a generalization of Zhang's theorem on successive minima.

2020 Mathematics Subject Classification: primary 14G40; secondary 11G50.
Keywords: adelic line bundles and divisors, arithmetic ampleness, Nakai-Moishezon criterion.

1. Introduction

In algebraic geometry, the Nakai-Moishezon criterion asserts that a Cartier divisor $D \in \mathrm{Div}(X)$ on a projective variety X over an algebraically closed field is ample if and only if $D^{\dim Y} \cdot Y > 0$ for every subvariety $Y \subseteq X$. By a theorem of Campana and Peternell [CP90], this statement remains valid when $D \in \mathrm{Div}(X)_{\mathbb{R}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is a $\mathbb{R}$-Cartier divisor. In [Zha95a], Zhang started the study of arithmetic ampleness in the context of Arakelov geometry, and proved an arithmetic Nakai–Moishezon criterion for adelic metrized line bundles ([Zha95a, Theorem 4.2]). Our purpose is to extend this result to adelic $\mathbb{R}$-Cartier divisors (in the sense of Moriwaki [Mor16]), thus proving an arithmetic analogue of Campana and Peternell's theorem.

Let X be a normal and geometrically integral projective scheme over a number field K. An adelic $\mathbb{R}$-Cartier divisor $\overline{D} = (D, (g_v)_v)$ on X is a pair consisting of a $\mathbb{R}$-Cartier divisor $D \in \mathrm{Div}(X)_{\mathbb{R}}$ and a suitable collection of Green functions $(g_v)_v$ on the analytifications X_v^{an} of X, where v runs over the set of places of K (see Definition 3.1). The set $\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ of adelic $\mathbb{R}$-Cartier divisors is a $\mathbb{R}$-vector space; it contains the set of adelic Cartier divisors $\widehat{\mathrm{Div}}(X)$, defined by

$$\widehat{\mathrm{Div}}(X) = \{(D, (g_v)_v) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}} \mid D \in \mathrm{Div}(X)\} \subseteq \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

To any adelic Cartier divisor $\overline{D} \in \widehat{\mathrm{Div}}(X)$ we can associate an adelic metrized line bundle $(\mathcal{O}_X(D), (\|.\|_v^{\overline{D}})_v)$ in the sense of Zhang [Zha95b], and a global section $s \in H^0(X, D)$ of $\mathcal{O}_X(D)$ is called strictly small if $\sup_{x \in X_v^{\mathrm{an}}} \|s\|_v^{\overline{D}}(x) \leq 1$ for every place v, with strict inequality at archimedean places. We say that an adelic $\mathbb{R}$-Cartier divisor $\overline{D}$ is ample if it is semi-positive (see Definition 3.4) and if it can be

Date: March 14, 2022.

written as a finite sum

$$\overline{D} = \sum_{i=1}^{\ell} \lambda_i \overline{A}_i$$

where for each $i \in \{1, \dots, \ell\}$, $\lambda_i \in \mathbb{R}_{>0}$ and $\overline{A}_i = (A_i, (g_{i,v})_v) \in \widehat{\mathrm{Div}}(X)$ is an adelic Cartier divisor such that $A_i \in \mathrm{Div}(X)$ is ample and $H^0(X, mA_i)$ has a K-basis consisting of strictly small sections for every $m \gg 1$. This definition of ampleness for adelic $\mathbb{R}$-Cartier divisors coincides with the one used in [BGMPS16] (see Remark 6.5). For any semi-positive $\overline{D} = (D, (g_v)_v) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and for any subvariety $Y \subseteq X$, we denote by $h_{\overline{D}}(Y)$ the height of Y with respect to $\overline{D}$ (see section 3.2). The main result in this paper is the following (see Corollary 6.4).

Theorem 1.1. *Let $\overline{D} = (D, (g_v)_v)$ be a semi-positive adelic $\mathbb{R}$-Cartier divisor on X. Then $\overline{D}$ is ample if and only if $h_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq X$.*

This theorem gives an affirmative answer to a question of Burgos Gil, Moriwaki, Philippon and Sombra [BGMPS16, Remark 3.21]. To our knowledge, it was known only under one of the following additional assumptions up to now:

- $\overline{D}$ is an adelic Cartier divisor (Zhang's arithmetic Nakai-Moishezon criterion [Zha95a, Theorem 4.2], [Mor15, Corollary 5.1], [CM18, Theorem 1.2]);
- $\overline{D}$ is a toric metrized $\mathbb{R}$-Cartier divisor ([BGMPS16, Corollary 6.3]);
- X has dimension one ([Iko21, Corollary A.4]).

Given a semi-positive adelic $\mathbb{R}$-Cartier divisor $\overline{D} = (D, (g_v)_v)$ on X and a subvariety $Y \subseteq X$ with $\deg_D(Y) := D^{\dim Y} \cdot Y \neq 0$, the normalized height of Y with respect to $\overline{D}$ is defined by

$$\widehat{h}_{\overline{D}}(Y) = \frac{h_{\overline{D}}(Y)}{(\dim Y + 1) \deg_D(Y)}.$$

We also let $\zeta_{\mathrm{abs}}(\overline{D}) = \inf_{x \in X(\overline{K})} \widehat{h}_{\overline{D}}(x)$. Our second main result is the following theorem, which plays an important role in this paper and might be of independent interest.

Theorem 1.2. *Let $\overline{D} = (D, (g_v)_v)$ be a semi-positive adelic $\mathbb{R}$-Cartier divisor on X. If D is ample, there exists a subvariety $Y \subseteq X$ such that*

$$\zeta_{\mathrm{abs}}(\overline{D}) = \widehat{h}_{\overline{D}}(Y) = \min_{Z \subseteq X} \widehat{h}_{\overline{D}}(Z),$$

where the minimum is over the subvarieties $Z \subseteq X$.

In other words, the infimum of the normalized heights of subvarieties $Z \subseteq X$ is attained at a subvariety Y, which moreover satisfies $\widehat{h}_{\overline{D}}(Y) = \zeta_{\mathrm{abs}}(\overline{D})$. Our proof of Theorem 1.2 is based on Zhang's theorem on successive minima [Zha95a, Theorem 5.2]. Although the latter does not appear in the literature for adelic $\mathbb{R}$-Cartier divisors, we shall prove that it remains valid in this context thanks to a continuity property for successive minima (see Lemma 4.1 and Theorem 4.3). This approach also provides additional information on the subvariety $Y \subseteq X$ of Theorem 1.2 (see Theorem 5.1). Our proof of Theorem 1.1 is very direct, and goes roughly as follows. Let $\overline{D} = (D, (g_v)_v) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be semi-positive, with D ample. Given a real number $t \in \mathbb{R}$, we define an adelic $\mathbb{R}$-Cartier divisor $\overline{D}(t)$ by rescaling the metrics at archimedean places to have $\widehat{h}_{\overline{D}(t)}(Y) = \widehat{h}_{\overline{D}}(Y) - t$ for every subvariety $Y \subseteq X$ (see Definition 3.3 and Lemma 3.7). In view of Theorem 1.2, it suffices to prove that

$$\sup\{t \in \mathbb{R} \mid \overline{D}(t) \text{ is ample}\} = \zeta_{\mathrm{abs}}(\overline{D}).$$

We denote by $\theta(\overline{D})$ the supremum on the left hand side. We first observe that Zhang's arithmetic Nakai-Moishezon criterion [Zha95a, Theorem 4.2] implies that

$\theta(\overline{D}) = \zeta_{\mathrm{abs}}(\overline{D})$ provided that $\overline{D}$ is an adelic Cartier divisor. We simply deduce the general case (Theorem 6.1) by slightly perturbing $\overline{D}$ and by applying a continuity property for the invariants $\zeta_{\mathrm{abs}}(\overline{D})$ and $\theta(\overline{D})$ (see Lemmas 4.1 and 6.2).

Organization of the paper. We fix some notation in section 2. In section 3 we recall the definition of adelic $\mathbb{R}$-Cartier divisors and of height of subvarieties. We also study some basic properties of ample adelic $\mathbb{R}$-Cartier divisors. We define successive minima in section 4, and we establish a continuity property allowing us to extend Zhang's theorem on minima to adelic $\mathbb{R}$-Cartier divisors (Lemma 4.1 and Theorem 4.3). We prove Theorem 1.2 in section 5 (Theorem 5.1) and Theorem 1.1 in section 6 (Corollary 6.4).

2. Conventions and terminology

2.1. We say that a scheme is integral if it is reduced and irreducible. Given a Noetherian integral scheme X, we denote by $\mathrm{Div}(X)$ the group of Cartier divisors on X and by $\mathrm{Rat}(X)$ the field of rational functions on X. If $\mathbb{K}$ denotes $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, we let $\mathrm{Div}(X)_{\mathbb{K}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{K}$. The elements of $\mathrm{Div}(X)_{\mathbb{K}}$ are called $\mathbb{K}$-Cartier divisors on X. If X is normal, we denote by $\mathrm{Supp}\, D$ the support of a $\mathbb{K}$-Cartier divisor D (see [Mor16, section 1.2] for details). It is a Zariski-closed subset of X. We let (ϕ) be the Cartier divisor associated to a rational function $\phi \in \mathrm{Rat}(X)^{\times}$.

2.2. Let X be a projective variety over a field K, i.e. X is an integral projective scheme on $\mathrm{Spec}\, K$. A subvariety $Y \subseteq X$ is an integral closed subscheme of X. Given an integer $r \in \{0, \dots, \dim X\}$, a r-cycle is a formal linear combination with integer coefficients of r-dimensional subvarieties in X. Given a $\mathbb{K}$-Cartier divisor D on X, we define the degree of a r-cycle Z with respect to D by $\deg_D(Z) = D^{\dim Z} \cdot Z$. In particular, if $x \in X(\overline{K})$ is a closed point (considered as a subvariety of X), then $\deg_D(\{x\}) = [K(x) : K]$ is the degree over K of the residue field $K(x)$ of $x \in X$.

2.3. Throughout this text, we fix a number field K and an algebraic closure $\overline{K}$ of K. We denote by Σ_K the set of places of K and by $\Sigma_{K,\infty} \subset \Sigma_K$ the set of archimedean places. For each $v \in \Sigma_K$, we let K_v be the completion of K with respect to v and we denote by $|.|_v$ the unique absolute value on K_v extending the usual absolute value $|.|_v$ on $\mathbb{Q}_v$: $|p|_v = p^{-1}$ if v is a non-archimedean place over a prime number p, and $|.|_v = |.|$ is the usual absolute value on $\mathbb{R}$ if v is archimedean.

2.4. Let X be a scheme on $\mathrm{Spec}\, K$. For each $v \in \Sigma_K$, we let $X_v = X \times_K \mathrm{Spec}\, K_v$ be the base change of X to K_v, and we denote by X_v^{an} the analytification of X_v in the sense of Berkovich (see [Mor16, section 1.3] for a short introduction). Given a closed point $x \in X_v$, we let $x^{\mathrm{an}} \in X_v^{\mathrm{an}}$ be the point corresponding to the unique absolute value on $K_v(x)$ extending $|.|_v$.

2.5. Let X be a normal projective variety on $\mathrm{Spec}\, K$. Let $D \in \mathrm{Div}(X)_{\mathbb{R}}$, $v \in \Sigma_K$ and let $D_v \in \mathrm{Div}(X_v)_{\mathbb{R}}$ be the pullback of D to X_v. We consider an open covering $X_v = \cup_{i=1}^{\ell} U_i$ such that D_v is defined by $f_i \in \mathrm{Rat}(X_v)^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$ on U_i for each $i \in \{1, \dots, \ell\}$. A continuous D-Green function on X_v^{an} is a function

$$g_v \colon X_v^{\mathrm{an}} \setminus (\mathrm{Supp}\, D_v)^{\mathrm{an}} \to \mathbb{R}$$

such that $g_v + \ln |f_i|_v^2$ extends to a continuous function on the analytification U_i^{an} of U_i for each $i \in \{1, \dots, \ell\}$. When v is archimedean, we say that g_v is smooth (respectively plurisubharmonic) if the extension of $g_v + \ln |f_i|_v^2$ to U_i^{an} is smooth (respectively plurisubharmonic) for each $i \in \{1, \dots, \ell\}$. We refer the reader to [Mor16, sections 1.4 and 2.1] for more details on Green functions.

2.6. Let X be a normal projective variety on $\operatorname{Spec} K$. Let $D \in \operatorname{Div}(X)_{\mathbb{K}}$ and let $U \subseteq \operatorname{Spec} \mathcal{O}_K$ be a non-empty open subset, where $\mathcal{O}_K$ is the ring of integers of K. A normal model $\mathcal{X}$ of X over U is an integral normal scheme $\mathcal{X}$ together with a projective dominant morphism $\pi_{\mathcal{X}}\colon \mathcal{X} \to U$ with generic fiber X. If $\mathcal{D}$ is a $\mathbb{K}$-Cartier divisor on $\mathcal{X}$ such that the restriction of $\mathcal{D}$ to X is equal to D, we say that $(\mathcal{X}, \mathcal{D})$ is a normal model of (X, D) over U. For each non-archimedean place $v \in U$, we denote by $g_{\mathcal{D},v}$ the D-Green function on X_v^{an} induced by $\mathcal{D}$ (see [Mor16, section 0.2] for details on this construction).

3. Adelic ℝ-Cartier divisors

In the remainder of the text, we consider a normal and geometrically integral projective variety X over the number field K. We define adelic $\mathbb{R}$-Cartier divisors in subsection 3.1. We then recall the notion of semi-positive adelic $\mathbb{R}$-Cartier divisors and we define heights of subvarieties in subsection 3.2. Subsection 3.3 contains basic facts concerning ample adelic $\mathbb{R}$-Cartier divisors.

3.1. Definitions.

In this paragraph, $\mathbb{K}$ denotes either $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.

Definition 3.1. An adelic $\mathbb{K}$-Cartier divisor on X is a pair $\overline{D} = (D, (g_v)_{v\in\Sigma_K})$ consisting of a $\mathbb{K}$-Cartier divisor D on X and of a continuous D-Green function g_v on X_v^{an} for each $v \in \Sigma_K$, satisfying the following condition: there exist a dense open subset U of $\operatorname{Spec} \mathcal{O}_K$ and a normal model $(\mathcal{X}, \mathcal{D})$ of (X, D) over U such that $g_v = g_{\mathcal{D},v}$ for all $v \in U$.

The set of adelic $\mathbb{K}$-Cartier divisors on X is a $\mathbb{K}$-module, denoted by $\widehat{\operatorname{Div}}(X)_{\mathbb{K}}$. Since X is normal, the natural map $\operatorname{Div}(X) \to \operatorname{Div}(X)_{\mathbb{K}}$ is injective. It follows that $\widehat{\operatorname{Div}}(X)_{\mathbb{Z}} \subset \widehat{\operatorname{Div}}(X)_{\mathbb{Q}} \subset \widehat{\operatorname{Div}}(X)_{\mathbb{R}}$. In the sequel, the elements of $\widehat{\operatorname{Div}}(X) := \widehat{\operatorname{Div}}(X)_{\mathbb{Z}}$ will be called adelic Cartier divisors for simplicity.

Let $\overline{D} = (D, (g_v)_{v\in\Sigma_K})$ be an adelic $\mathbb{R}$-Cartier divisor on X. We consider the K-vector space

$$H^0(X, D) := \{\phi \in \operatorname{Rat}(X)^\times \mid D + (\phi) \geq 0\} \cup \{0\}.$$

For any $\phi \in (\operatorname{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}) \cup \{0\}$ and any $v \in \Sigma_K$, we let ϕ_v be the pullback of ϕ on X_v^{an} and we consider the function $\|\phi\|_v^{\overline{D}} := |\phi_v|_v \exp(-g_v/2)$, defined on an open subset of X_v^{an}. If $\phi \in H^0(X, D)$, the function $\|\phi\|_v^{\overline{D}}$ extends to a continuous function on X_v^{an} (see [Mor16, Propositions 1.4.2 and 2.1.3]). In that case, we let $\|\phi\|_{v,\sup}^{\overline{D}} := \sup_{x\in X_v^{\mathrm{an}}} \|\phi\|_v^{\overline{D}}(x)$. We also define the set of strictly small sections of $\overline{D}$ by

$$\widehat{H}^0(X, \overline{D}) := \{\phi \in H^0(X, D) \mid \|\phi\|_{v,\sup}^{\overline{D}} \leq 1 \ \forall v \in \Sigma_K, \quad \|\phi\|_{v,\sup}^{\overline{D}} < 1 \ \forall v \in \Sigma_{K,\infty}\}.$$

Remark 3.2. Let $\overline{D} \in \widehat{\operatorname{Div}}(X)$ be an adelic Cartier divisor. With the above notation, the pair $(\mathcal{O}_X(D), (\|.\|_v^{\overline{D}})_{v\in\Sigma_K})$ is an adelic metrized line bundle in the sense of Zhang [Zha95b, (1.2)]. One can see that every adelic metrized line bundle $\overline{L} = (L, (\|.\|_v)_{v\in\Sigma_K})$ on X can be obtained in this way by considering the Cartier divisor $D = \operatorname{div}(s)$ associated to a trivialization s of L and the D-Green functions $g_v = -\ln \|s_v\|_v^2$ for every $v \in \Sigma_K$, where s_v is the pullback of s to X_v^{an}.

We end this paragraph with the definition of twists of adelic $\mathbb{R}$-Cartier divisors, which we shall use frequently in the rest of the text.

Definition 3.3. Let $\overline{D} \in \widehat{\operatorname{Div}}(X)_{\mathbb{R}}$. For any real number $t \in \mathbb{R}$, we define the t-twist of $\overline{D}$ by

$$\overline{D}(t) = \overline{D} - t\overline{\xi}_\infty \in \widehat{\operatorname{Div}}(X)_{\mathbb{R}},$$

where $\overline{\xi}_\infty = (0, (\xi_v)_{v\in\Sigma_K})$ is the adelic Cartier divisor on X given by $\xi_v = 2$ if v is archimedean, and $\xi_v = 0$ otherwise.

It follows from the definitions that for any $\phi \in H^0(X, D)$, we have $\|\phi\|_v^{\overline{D}(t)} = e^t\|\phi\|_v^{\overline{D}}$ for every $v \in \Sigma_{K,\infty}$ and $\|\phi\|_v^{\overline{D}(t)} = \|\phi\|_v^{\overline{D}}$ for every $v \in \Sigma_K \setminus \Sigma_{K,\infty}$.

3.2. **Semi-positivity and heights of subvarieties.** Let us first define the height of a point $x \in X(\overline{K})$ with respect to an adelic ℝ-Cartier divisor $\overline{D}$ on X. Let $\phi \in \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$ be a function with $x \notin \mathrm{Supp}(D + (\phi))$ and let $K(x)$ be the function field of $x \in X$. For each place $w \in \Sigma_{K(x)}$, we fix a K-embedding $\sigma_w \colon K(x) \hookrightarrow \overline{K}_v$, where v denotes the restriction of w to K (note that there are exactly $[K(x)_w : K_v]$ such embeddings). The pair (x, σ_w) determines uniquely a point $x_w \in X_v$, and the quantity $\|\phi\|_w^{\overline{D}}(x) := \|\phi\|_v^{\overline{D}}(x_w^{\mathrm{an}})$ does not depend on the choice of σ_w. The normalized height of x with respect to $\overline{D}$ is the real number

$$\widehat{h}_{\overline{D}}(x) = -\sum_{w\in\Sigma_{K(x)}} \frac{[K(x)_w : \mathbb{Q}_w]}{[K(x) : \mathbb{Q}]} \ln \|\phi\|_w^{\overline{D}}(x).$$

This definition does not depend on the choice of ϕ by [Mor16, (4.2.1)]. Moreover, if $\phi \in H^0(X, D) \setminus \{0\}$ then it follows from the definitions that

$$\widehat{h}_{\overline{D}}(x) \geq -\sum_{v\in\Sigma_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \ln \|\phi\|_{v,\sup}^{\overline{D}}. \tag{3.1}$$

In order to define the height of higher dimensional subvarieties, we need the notion of semi-positive adelic ℝ-Cartier divisors which we recall below.

Definition 3.4. Let $\overline{D} = (D, (g_v)_{v\in\Sigma_K}) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. We say that $\overline{D}$ is semi-positive if there exists a sequence $(\mathcal{X}_n, \mathcal{D}_n, (g_{n,v})_{v\in\Sigma_K})_{n\in\mathbb{N}}$ such that :

- for all $n \in \mathbb{N}$, $(\mathcal{X}_n, \mathcal{D}_n)$ is a normal $\mathrm{Spec}\,\mathcal{O}_K$-model for (X, D) with $\mathcal{D}_n$ relatively nef,
- for all $n \in \mathbb{N}$, $g_{n,v}$ is a smooth plurisubharmonic D-Green function if $v \in \Sigma_{K,\infty}$ and $g_{n,v} = g_{\mathcal{D}_n,v}$ for every non-archimedean $v \in \Sigma_K$,
- for every $v \in \Sigma_K$, $(g_{n,v})_{n\in\mathbb{N}}$ converges uniformly to g_v.

Remark 3.5.

(1) It follows from the definition that the sum of semi-positive adelic ℝ-Cartier divisors is semi-positive. Moreover, if $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ is semi-positive then $\overline{D}(t)$ is semi-positive for any $t \in \mathbb{R}$.

(2) An adelic Cartier divisor $\overline{D} \in \widehat{\mathrm{Div}}(X)$ is semi-positive if and only if the associated line bundle $(\mathcal{O}_X(D), (\|.\|_v^{\overline{D}})_{v\in\Sigma_K})$ of Remark 3.2 is semi-positive in the sense of Zhang [Zha95b, (1.3)] (see [BGMPS16], (1) page 229).

Following [BGMPS16], we say that an adelic ℝ-Cartier divisor $\overline{D}$ on X is DSP if $\overline{D} = \overline{D}_1 - \overline{D}_2$ is the difference of two semi-positive $\overline{D}_1, \overline{D}_2 \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Let $\overline{D}$ be a DSP adelic ℝ-Cartier divisor on X and let $Y \subseteq X$ be a r-dimensional subvariety, where $0 \leq r \leq \dim X$ is an integer. For any place $v \in \Sigma_K$, we define a measure $c_1(\overline{D})^{\wedge \dim Y} \wedge \delta_{Y_v^{\mathrm{an}}}$ on X_v^{an} as in [BGMPS16, page 225]. It is obtained by multilinearity from the corresponding measures associated to semi-positive adelic Cartier divisors defined in [BGPS14, Definition 1.4.6]. The measure $c_1(\overline{D})^{\wedge \dim Y} \wedge \delta_{Y_v^{\mathrm{an}}}$ is supported on $Y_v^{\mathrm{an}} \subseteq X_v^{\mathrm{an}}$ and has total mass $\deg_D(Y)$.

Let $\Phi = (\phi_0, \ldots, \phi_r) \in (\mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R})^{\oplus r}$ be a family intersecting Y properly in the following sense: for every $I \subseteq \{0, \ldots, r\}$,

$$Y \cap \left(\bigcap_{i\in I} \mathrm{Supp}((\phi_i) + D) \right)$$

is of pure dimension $r - \#I$. The local height $h_{\overline{D},\Phi,v}(Y)$ of Z at v with respect to $(\overline{D},\Phi)$ is defined inductively as follows. We put $h_{\overline{D},\Phi,v}(\emptyset)=0$, and

$$h_{\overline{D},\Phi,v}(Z) = h_{\overline{D},(\phi_1,\ldots,\phi_r),v}(Y\cdot(D+(\phi_0))) - \int_{X_v^{\mathrm{an}}} \ln\|\phi_0\|_v^{D} c_1(\overline{D})^{\wedge \dim Y}\wedge\delta_{Y_v^{\mathrm{an}}}. \tag{3.2}$$

It follows from [BGPS14, Proposition 1.5.14] that $h_{\overline{D},\Phi,v}(Y)=0$ for all except finitely many places $v\in\Sigma_K$. The height of Y with respect to $\overline{D}$ is the real number

$$h_{\overline{D}}(Y) = \sum_{v\in\Sigma_K}\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}h_{\overline{D},\Phi,v}(Y);$$

it does not depend on the choice of Φ. If $Y\subseteq X$ is a subvariety with $\deg_D(Y)\neq 0$, the normalized height of Y with respect to $\overline{D}$ is the real number

$$\widehat{h}_{\overline{D}}(Y) = \frac{h_{\overline{D}}(Y)}{(\dim Y+1)\deg_D(Y)}.$$

Remark 3.6.

(1) If $Y=\{x\}$ is a closed point in X, then $\widehat{h}_{\overline{D}}(Y)$ coincides with the normalized height $\widehat{h}_{\overline{D}}(x)$ of x.

(2) The height function is continuous in the following sense: for any DSP adelic $\mathbb{R}$-Cartier divisor $\overline{D}'$ on X, we have

$$\lim_{t\to 0} h_{\overline{D}+t\overline{D}'}(Y) = h_{\overline{D}}(Y).$$

If moreover $\deg_D(Y)\neq 0$, then $\deg_{D+tD'}(Y)\neq 0$ for any sufficiently small $t\in\mathbb{R}$ and we have $\lim_{t\to 0}\widehat{h}_{\overline{D}+t\overline{D}'}(Y)=\widehat{h}_{\overline{D}}(Y)$.

(3) Assume that $\overline{D}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ is semi-positive, and let

$$(\mathcal{X}_n,\mathcal{D}_n,(g_{n,v})_{v\in\Sigma_K})_{n\in\mathbb{N}}$$

be a sequence as in Definition 3.4. Given $n\in\mathbb{N}$, let $\overline{D}_n=(D,(g_{n,v})_{v\in\Sigma_K})$. Then we have $\lim_{n\to\infty}h_{\overline{D}_n}(Y)=h_{\overline{D}}(Y)$, and moreover $\lim_{n\to\infty}\widehat{h}_{\overline{D}_n}(Y)=\widehat{h}_{\overline{D}}(Y)$ if $\deg_D(Y)\neq 0$.

(4) Assume that $\overline{D}=(D,(g_v)_{v\in\Sigma_K})\in\widehat{\mathrm{Div}}(X)$ is a semi-positive adelic Cartier divisor such that there exists a $\operatorname{Spec}\mathcal{O}_K$-model $(\mathcal{X},\mathcal{D})$ of (X,D) with $g_v=g_{\mathcal{D},v}$ for every non-archimedean place $v\in\Sigma_K$. Then

$$\overline{\mathcal{L}} = (\mathcal{O}_{\mathcal{X}}(\mathcal{D}),(\|.\|_v^{D})_{v\in\Sigma_{K,\infty}})$$

is a semi-positive hermitian line bundle in the sense of [Zha95a] and we have $h_{\overline{D}}(Y)=c_1(\overline{\mathcal{L}}_{|\mathcal{Y}})^{\dim\mathcal{Y}}$, where $\mathcal{Y}$ is the Zariski-closure of Y in $\mathcal{X}$ (see [Zha95a, (1.2)] for the definition of $c_1(\overline{\mathcal{L}}_{|\mathcal{Y}})^{\dim\mathcal{Y}}$).

We have the following lemma concerning the behaviour of heights with respect to twists of adelic $\mathbb{R}$-Cartier divisors (see Definition 3.3).

Lemma 3.7. *Let $\overline{D}=(D,(g_v)_{v\in\Sigma_K})$ be a DSP adelic $\mathbb{R}$-Cartier divisor on X and let $Y\subseteq X$ be a subvariety. For any $t\in\mathbb{R}$, we have*

$$h_{\overline{D}(t)}(Y) = h_{\overline{D}}(Y) - t(\dim Y+1)\deg_D(Y).$$

In particular, if $\deg_D(Y)\neq 0$ *then* $\widehat{h}_{\overline{D}(t)}(Y)=\widehat{h}_{\overline{D}}(Y)-t$.

Proof. The result follows from (3.2) by induction on $\dim Y$. □

Let $r \in \{0, \ldots, \dim X\}$ and let Z be a r-cycle in $X_{\overline{K}} = X \times_K \operatorname{Spec} \overline{K}$. There exists a finite extension K' of K such that Z is defined over K', i.e. Z is a r-cycle in $X_{K'} = X \times_K \operatorname{Spec} K'$: there exists integers $a_1, \ldots, a_\ell$ and subvarieties $Y_1, \ldots, Y_\ell$ of $X_{K'}$ such that $Z = \sum_{i=1}^{\ell} a_i Y_{i,\overline{K}}$. Given a DSP $\overline{D} \in \widehat{\operatorname{Div}}(X)_{\mathbb{R}}$, we define a DSP adelic $\mathbb{R}$-Cartier divisor $\overline{D}_{K'}$ by pulling back $\overline{D}$ to $X_{K'}$. The height of Z with respect to $\overline{D}$ is then defined by $h_{\overline{D}}(Z) = \sum_{i=1}^{\ell} a_i h_{\overline{D}_{K'}}(Y_i)$. This definition does not depend on the choice of K' by [BGPS14, Proposition 1.5.10].

Lemma 3.8. *Let $\overline{D}$ be a DSP adelic $\mathbb{R}$-Cartier divisor on X. The following conditions are equivalent:*

(1) $h_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq X$;
(2) $h_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq X_{\overline{K}}$.

Proof. The implication (2) $\Longrightarrow$ (1) is clear. Assume that (1) holds and let $Y \subseteq X_{\overline{K}}$ be a subvariety. Let $\operatorname{Gal}(\overline{K}/K)$ be the set of K-automorphisms $\sigma\colon \overline{K} \to \overline{K}$. For any $\sigma \in \operatorname{Gal}(\overline{K}/K)$, we denote by Y^σ the pullback of Y by the automorphism of $X_{\overline{K}}$ induced by σ; it is a subvariety of $X_{\overline{K}}$. We consider the set

$$O(Y) = \{Y^\sigma \mid \sigma \in \operatorname{Gal}(\overline{K}/K)\}.$$

It follows easily from the definitions that $h_{\overline{D}}(Y') = h_{\overline{D}}(Y)$ for any $Y' \in O(Y)$ (alternatively, this fact is a direct consequence of [BGPS14, Theorem 1.5.11]). By [BG06, A.4.13],

$$Z_Y = \bigcup_{Y' \in O(Y)} Y'$$

is a subvariety of X (i.e. its image in X is an irreducible Zariski closed subset of X, which we still denote by Z_Y). Therefore $h_{\overline{D}}(Z_Y) > 0$ by assumption. Let K' be a finite extension such that every $Y' \in O(Y)$ is a subvariety of $X_{K'}$. Let $(Z_Y)_{K'}$ be the cycle in $X_{K'}$ associated to Z_Y: we have

$$(Z_Y)_{K'} = \sum_{Y' \in O(Y)} n_{Y'} Y',$$

where $n_{Y'}$ is a positive integer for every $Y' \in O(Y)$. By [BGPS14, Proposition 1.5.10], we have $h_{\overline{D}}((Z_Y)_{K'}) = h_{\overline{D}}(Z_Y) > 0$. On the other hand, we have

$$h_{\overline{D}}((Z_Y)_{K'}) = \sum_{Y' \in O(Y)} n_{Y'} h_{\overline{D}}(Y') = h_{\overline{D}}(Y) \times \sum_{Y' \in O(Y)} n_{Y'},$$

and therefore $h_{\overline{D}}(Y) > 0$. □

We end this paragraph with a sufficient condition for the ampleness of the underlying divisor of an adelic $\mathbb{R}$-Cartier divisor.

Lemma 3.9. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\operatorname{Div}}(X)_{\mathbb{R}}$ be semi-positive. Assume that $h_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq X$. Then D is ample.*

We want to combine Campana and Peternell's Nakai-Moishezon criterion for $\mathbb{R}$-Cartier divisors [CP90] with Moriwaki's generalized Hodge index theorem [Mor16, Theorem 5.3.2] applied to subvarieties of X. We must pay attention to the fact that [Mor16, Theorem 5.3.2] applies only to normal and geometrically integral subvarieties.

Proof. Let $Y \subseteq X_{\overline{K}}$ be a subvariety and let K' be a finite extension of K such that Y is defined over K'. We consider the adelic $\mathbb{R}$-Cartier divisor $\overline{D}_{K'} = (D_{K'}, (g_w)_{w \in \Sigma_{K'}})$ defined by pulling back $\overline{D}$ to $X_{K'}$. Let $f\colon Y' \to Y$ be the normalization of Y and let $\phi \in \operatorname{Rat}(X_{K'})^\times \otimes_{\mathbb{Z}} \mathbb{R}$ be such that $Y \nsubseteq \operatorname{Supp}(D_{K'} + (\phi))$. Note that Y' is normal and geometrically integral. We define a semi-positive adelic $\mathbb{R}$-Cartier divisor $\overline{D}_{Y'} = (D_{Y'}, (g_{Y',w})_{w\in\Sigma_{K'}})$ on Y' as follows: $D_{Y'} = f^*(D_{K'} + (\phi))_{|Y}$ and for each $w \in \Sigma_{K'}$, the $D_{Y'}$-Green function $g_{Y',w}$ is the pull-back of $(g_w - 2\ln|\phi|_w)_{|Y_w^{\mathrm{an}}}$ to $(Y'_w)^{\mathrm{an}}$. By [BGPS14, Theorem 1.5.11 (2)], we have $h_{\overline{D}_{Y'}}(Y') = h_{\overline{D}_{K'}}(Y)$. Therefore our assumption together with Lemma 3.8 implies that $h_{\overline{D}_{Y'}}(Y') = h_{\overline{D}_{K'}}(Y) > 0$. It follows from [Mor16, Theorem 5.3.2] that $\overline{D}_{Y'}$ is big in the sense of [Mor16, Definition 4.4.1]. In particular, $D_{Y'}$ is big. Since $D_{Y'}$ is also nef by semi-positivity, we have $D_{K'}^{\dim Y} \cdot Y = D_{Y'}^{\dim Y'} \cdot Y' > 0$. Therefore D is ample by [CP90, Theorem 1.3]. □

3.3. **Ample adelic ℝ-Cartier divisors.** We now define ample adelic $\mathbb{R}$-Cartier divisors and study some of their properties.

Definition 3.10. Let $\overline{D} = (D, (g_v)_{v\in\Sigma_K})$ be an adelic $\mathbb{R}$-Cartier divisor. We say that $\overline{D}$ is

- weakly ample (w-ample for short) if $\overline{D} = \sum_{i=1}^{\ell} \lambda_i \overline{A}_i$ is a $\mathbb{R}$-linear combination of adelic Cartier divisors $\overline{A}_i \in \widehat{\mathrm{Div}}(X)$ such that for each $i \in \{1,\dots,\ell\}$, $\lambda_i > 0$, A_i is ample and for every $m \gg 1$, $H^0(X, mA_i)$ has a K-basis consisting of strictly small sections;
- ample if it is w-ample and semi-positive.

The terminology of weakly ample adelic $\mathbb{R}$-Cartier divisors is due to Ikoma [Iko21]. We end this section with three lemmas concerning basic properties of w-ample adelic $\mathbb{R}$-Cartier divisors.

Lemma 3.11. *Let $\overline{D}, \overline{D}' \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. If $\overline{D}$ is w-ample, there exists a real number $\varepsilon > 0$ such that $\overline{D} + t\overline{D}'$ is w-ample for any $t \in \mathbb{R}$ with $|t| \leq \varepsilon$.*

Proof. Without loss of generality, we only consider the case where $\overline{D}' \in \widehat{\mathrm{Div}}(X)$ and $t \geq 0$. If $\overline{D}$ is w-ample, $\overline{D} = \sum_{i=1}^{\ell} \lambda_i \overline{A}_i$ is a $\mathbb{R}$-linear combination with positive coefficients of adelic Cartier divisors $\overline{A}_i \in \widehat{\mathrm{Div}}(X)$ such that for each $i \in \{1,\dots,\ell\}$, A_i is ample and $H^0(X, mA_i)$ has a K-basis consisting of strictly small sections for $m \gg 1$. By [Iko16, Proposition 5.4 (5)], there exists a $\delta > 0$ such that $\overline{A}_1 + \delta\overline{D}'$ is w-ample. Let $\varepsilon = \delta\lambda_1$. Then for every real number $t \in [0, \varepsilon]$,

$$\overline{D} + t\overline{D}' = \frac{t}{\delta}(\overline{A}_1 + \delta\overline{D}') + (\lambda_1 - \frac{t}{\delta})\overline{A}_1 + \sum_{i=2}^{\ell} \lambda_i \overline{A}_i$$

is w-ample. □

Remark 3.12. By Lemma 3.11 and [Mor16, Lemma 1.1.1], an adelic Cartier divisor $\overline{D} = (D, (g_v)_{v\in\Sigma_K}) \in \widehat{\mathrm{Div}}(X)$ on X is w-ample if and only if D is ample and $H^0(X, mD)$ has a K-basis consisting of strictly small sections for every $m \gg 1$.

Lemma 3.13. *Let $\overline{D}$ be a w-ample adelic $\mathbb{R}$-Cartier divisor on X. Then*

$$\inf_{x\in X(\overline{K})} \widehat{h}_{\overline{D}}(x) > 0.$$

Proof. By definition, we can write $\overline{D} = \sum_{i=1}^{\ell} \lambda_i \overline{A}_i$ where for each $i \in \{1,\dots,\ell\}$, λ_i is a positive real number, $\overline{A}_i$ is an adelic Cartier divisor such that A_i is ample, and $H^0(X, mA_i)$ has a K-basis consisting of strictly small sections for every $m \gg 1$. Let $m \geq 1$ be an integer such that for each $i \in \{1,\dots,\ell\}$, there exists a set of functions $\phi_{i,1},\dots,\phi_{i,k_i} \in \widehat{H}^0(X, mA_i)$ with

$$\bigcap_{j=1}^{k_i} \mathrm{Supp}(mA_i + (\phi_{i,j})) = \emptyset.$$

Letting

$$\Lambda_i := -\max_{1\leq j\leq k_i}\sum_{v\in\Sigma_K}\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\ln\|\phi_{i,j}\|_{v,\sup}^{m\overline{A}_i} > 0,$$

we have $\widehat{h}_{\overline{A}_i}(x)\geq \Lambda_i/m$ for every $x\in X(\overline{K})$ (see (3.1)). Therefore we have

$$\inf_{x\in X(\overline{K})}\widehat{h}_{\overline{D}}(x)\geq \sum_{i=1}^{\ell}\lambda_i\inf_{x\in X(\overline{K})}\widehat{h}_{\overline{A}_i}(x)\geq \sum_{i=1}^{\ell}\lambda_i\Lambda_i/m>0.$$

□

Lemma 3.14. *Let $\overline{D}=(D,(g_v)_{v\in\Sigma_K})$ be an adelic $\mathbb{R}$-Cartier divisor. If D is ample, there exists a real number $t\in\mathbb{R}$ such that $\overline{D}(t)$ is w-ample.*

Proof. Since D is ample, there exists an ample $\overline{A}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ such that $\overline{D}-\overline{A}\in\widehat{\mathrm{Div}}(X)_{\mathbb{Q}}$ and $D-A$ is ample. For a sufficiently large and divisible integer m, $m(D-A)$ is a very ample Cartier divisor on X. Let $(\phi_1,\dots,\phi_\ell)$ be basis of $H^0(X,m(D-A))$ such that $\|\phi_i\|_{v,\sup}^{m(\overline{D}-\overline{A})}\leq 1$ for every $i\in\{1,\dots,\ell\}$ and every non-archimedean place $v\in\Sigma_K$. Let $t\in\mathbb{R}$ be a real number such that

$$t<-\max_{1\leq i\leq\ell}\max_{v\in\Sigma_{K,\infty}}\ln\|\phi_i\|_{v,\sup}^{m(\overline{D}-\overline{A})}.$$

Then $\phi_i\in\widehat{H}^0(X,m(\overline{D}-\overline{A})(t))$ for every i, and it follows that $\overline{A}'_t:=(\overline{D}-\overline{A})(t)=\overline{D}(t)-\overline{A}$ is w-ample. Therefore $\overline{D}(t)=\overline{A}+\overline{A}'_t$ is w-ample. □

4. Zhang's theorem on successive minima

In this section we recall the notion of successive minima for adelic $\mathbb{R}$-Cartier divisors, which was first introduced by Zhang for hermitian line bundles [Zha95a, section 5]. We then prove a continuity property which allows to extend Zhang's theorem on minima [Zha95a, Theorem 5.2] to the case of adelic $\mathbb{R}$-Cartier divisors (see Lemma 4.1 and Theorem 4.3 below).

Let $\overline{D}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and let $Z\subseteq X$ be a subvariety. For any $i\in\{1,\dots,\dim Z+1\}$, we define the i-th successive minimum of $\overline{D}$ on Z by

$$\zeta_i(\overline{D},Z)=\sup_{\substack{Y\subsetneq Z\\ \dim Y<i-1}}\inf_{x\in Z(\overline{K})\setminus Y}\widehat{h}_{\overline{D}}(x)\in\mathbb{R}\cup\{-\infty\},$$

where the supremum is over all the Zariski-closed subsets $Y\subsetneq Z$ of dimension $\dim Y<i-1$. We obtain a chain of real numbers

$$\zeta_{\dim Z+1}(\overline{D},Z)\geq\zeta_{\dim Z}(\overline{D},Z)\geq\cdots\geq\zeta_1(\overline{D},Z).$$

Successive minima satisfy the following properties.

Lemma 4.1. *Let $\overline{D}=(D,(g_v)_{v\in\Sigma_K})\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Let $Z\subseteq X$ be a subvariety and let $1\leq i\leq\dim Z+1$ be an integer.*

(1) For any $\overline{D}'\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, we have

$$\zeta_i(\overline{D}+\overline{D}',Z)\geq\zeta_i(\overline{D},Z)+\zeta_i(\overline{D}',Z).$$

(2) Let $\overline{D}_1,\dots,\overline{D}_\ell\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. If D is ample, then

$$\lim_{\max\{|t_1|,\dots,|t_\ell|\}\to 0}\zeta_i(\overline{D}+t_1\overline{D}_1+\cdots+t_\ell\overline{D}_\ell,Z)=\zeta_i(\overline{D},Z).$$

Proof. (1) We may assume that $\zeta_i(\overline{D}, Z) > -\infty$ and $\zeta_i(\overline{D}', Z) > -\infty$. Let $t < \zeta_i(\overline{D}, Z)$ and $t' < \zeta_i(\overline{D}', Z)$ be real numbers. By definition, there exist two closed subsets $Y, Y' \subseteq Z$ of dimension $< i-1$ such that for any $x \in Z(\overline{K}) \setminus (Y \cup Y')$, we have

$$\widehat{h}_{\overline{D}+\overline{D}'}(x) = \widehat{h}_{\overline{D}}(x) + \widehat{h}_{\overline{D}'}(x) \geq t + t'.$$

Since $\dim(Y \cup Y') < i - 1$, we have

$$\zeta_i(\overline{D} + \overline{D}', Z) \geq \inf_{x \in Z(\overline{K}) \setminus (Y \cup Y')} \widehat{h}_{\overline{D}}(x) \geq t + t',$$

and we conclude by letting t and t' tend to $\zeta_i(\overline{D}, Z)$ and $\zeta_i(\overline{D}', Z)$.

(2) If we replace $\overline{D}$ by $\overline{D}(t)$ for some real number t, both sides of the equality differ by $-t$. By Lemma 3.14, we may therefore assume that $\overline{D}$ is w-ample. Let $\varepsilon > 0$ be a real number. For $t_1, \ldots, t_\ell \in \mathbb{R}$ small enough, the adelic $\mathbb{R}$-Cartier divisors

$$(1+\varepsilon)\overline{D} - (\overline{D} + t_1\overline{D}_1 + \cdots + t_\ell\overline{D_\ell}) = \varepsilon\overline{D} - (t_1\overline{D}_1 + \cdots + t_\ell\overline{D_\ell})$$

and

$$\overline{D} + t_1\overline{D}_1 + \cdots + t_\ell\overline{D_\ell} - (1-\varepsilon)\overline{D} = \varepsilon\overline{D} + (t_1\overline{D}_1 + \cdots + t_\ell\overline{D_\ell})$$

are w-ample by Lemma 3.11. Combining (1) and Lemma 3.13, we have

$$(1+\varepsilon)\zeta_i(\overline{D}, Z) \geq \zeta_i(\overline{D} + t_1\overline{D}_1 + \cdots + t_\ell\overline{D}_\ell, Z) \geq (1-\varepsilon)\zeta_i(\overline{D}, Z)$$

and the result follows. □

Remark 4.2. Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be semi-positive. We consider a sequence $(\mathcal{X}_n, \mathcal{D}_n, (g_{n,v})_{v \in \Sigma_K})_{n \in \mathbb{N}}$ associated to $\overline{D}$ as in Definition 3.4. For each $n \in \mathbb{N}$, let $\overline{D}_n = (D, (g_{n,v})_{v \in \Sigma_K})$. Then we have

$$\lim_{n \to \infty} \zeta_i(\overline{D}_n, Z) = \zeta_i(\overline{D}, Z)$$

for any subvariety $Z \subseteq X$ and any $i \in \{1, \ldots, \dim Z + 1\}$. Indeed, the sum

$$\varepsilon_n := 2 \sum_{v \in \Sigma_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \sup_{z \in X_v^{\mathrm{an}}} |g_v(z) - g_{n,v}(z)|$$

is finite for every $n \in \mathbb{N}$, and the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ converges to zero. By construction, we have

$$\widehat{h}_{\overline{D}_n}(x) - \varepsilon_n \leq \widehat{h}_{\overline{D}}(x) \leq \widehat{h}_{\overline{D}_n}(x) + \varepsilon_n.$$

for any $n \in \mathbb{N}$ and $x \in X(\overline{K})$. It follows that

$$\zeta_i(\overline{D}_n, Z) - \varepsilon_n \leq \zeta_i(\overline{D}, Z) \leq \zeta_i(\overline{D}_n, Z) + \varepsilon_n$$

as in the proof of Lemma 4.1 (1), and we conclude by letting n tend to infinity.

The following theorem was originally proved by Zhang for adelic Cartier divisors equipped with Green functions induced by a fixed model [Zha95a, Theorem 5.2]. Thanks to the continuity property of Lemma 4.1, it remains valid for adelic $\mathbb{R}$-Cartier divisors.

Theorem 4.3. *Assume that $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ is semi-positive and that D is ample. For any subvariety $Z \subseteq X$, we have*

$$\zeta_{\dim Z+1}(\overline{D}, Z) \geq \widehat{h}_{\overline{D}}(Z) \geq \frac{1}{\dim Z + 1} \sum_{i=1}^{\dim Z+1} \zeta_i(\overline{D}, Z).$$

Proof. Since D is ample, we can write $D = \sum_{i=1}^{\ell} \lambda_i A_i$ where for each $i \in \{1, \ldots, \ell\}$, $\lambda_i \in \mathbb{R}_{>0}$ and $A_i \in \mathrm{Div}(X)$ is an ample Cartier divisor on X. Let $(g_{i,v})_{v\in\Sigma_K}$ be a collection of A_i-Green functions such that $\overline{A}_i = (A_i, (g_{i,v})_{v\in\Sigma_K})$ is a semi-positive adelic Cartier divisor on X. Given a ℓ-tuple of real numbers $\mathbf{t} = (t_1, \ldots, t_\ell) \in \mathbb{R}^\ell$, we denote by $\overline{D}_{\mathbf{t}} = (D_{\mathbf{t}}, (g_{\mathbf{t},v})_{v\in\Sigma_K})$ the adelic $\mathbb{R}$-Cartier divisor

$$\overline{D}_{\mathbf{t}} = \overline{D} + \sum_{i=1}^{\ell} t_i \overline{A}_i = (\sum_{i=1}^{\ell} (\lambda_i + t_i) A_i, (g_v + \sum_{i=1}^{\ell} t_i g_{i,v})_{v\in\Sigma_K}).$$

Let $\varepsilon > 0$ be a real number. We can choose $\mathbf{t} \in [0, \varepsilon]^\ell$ such that $\overline{D}_{\mathbf{t}} \in \widehat{\mathrm{Div}}(X)_{\mathbb{Q}}$. Note that $\overline{D}_{\mathbf{t}} \in \widehat{\mathrm{Div}}(X)_{\mathbb{Q}}$ is semi-positive. We consider a sequence

$$(\mathcal{X}_{\mathbf{t},n}, \mathcal{D}_{\mathbf{t},n}, (g_{\mathbf{t},n,v})_{v\in\Sigma_K})_{n\in\mathbb{N}}$$

associated to $\overline{D}_{\mathbf{t}}$ as in Definition 3.4, and we let $\overline{D}_{\mathbf{t},n} = (D_{\mathbf{t}}, (g_{\mathbf{t},n,v})_{v\in\Sigma_K}) \in \widehat{\mathrm{Div}}(X)_{\mathbb{Q}}$. Let m be a positive integer such that $mD_{\mathbf{t},n} \in \mathrm{Div}(X)$. By [Mor15, Theorem 0.2], the hermitian metrized line bundle $\overline{\mathcal{L}}_{m,\mathbf{t},n}$ associated to $m\overline{D}_{\mathbf{t},n}$ in Remark 3.6 (4) is semiample metrized in the sense of [Zha95a, section 5]. Therefore we can apply [Zha95a, Theorem 5.2] to the restriction of $\overline{\mathcal{L}}_{m,\mathbf{t},n}$ to the closure of Z in $\mathcal{X}_{\mathbf{t},n}$. We obtain

$$\zeta_{\dim Z+1}(m\overline{D}_{\mathbf{t},n}, Z) \geq \widehat{h}_{m\overline{D}_{\mathbf{t},n}}(Z) \geq \frac{1}{\dim Z + 1} \sum_{i=1}^{\dim Z+1} \zeta_i(m\overline{D}_{\mathbf{t},n}, Z) \tag{4.1}$$

for any $n \in \mathbb{N}$ (see Remark 3.6 (4)). On the other hand we have $\widehat{h}_{m\overline{D}_{\mathbf{t},n}}(Z) = m\widehat{h}_{\overline{D}_{\mathbf{t},n}}(Z)$ and $\zeta_i(m\overline{D}_{\mathbf{t},n}, Z) = m\zeta_i(\overline{D}_{\mathbf{t},n}, Z)$ for any $i \in \{1, \ldots, \dim Z + 1\}$, and therefore (4.1) remains true for $m = 1$. Letting n tend to infinity, we obtain

$$\zeta_{\dim Z+1}(\overline{D}_{\mathbf{t}}, Z) \geq \widehat{h}_{\overline{D}_{\mathbf{t}}}(Z) \geq \frac{1}{\dim Z + 1} \sum_{i=1}^{\dim Z+1} \zeta_i(\overline{D}_{\mathbf{t}}, Z)$$

by Remarks 3.6 (3) and 4.2. Letting ε tend to zero, the result follows from the continuity of normalized heights and successive minima given by Remark 3.6 (2) and Lemma 4.1 (2). □

5. Absolute minimum and height of subvarieties

For any $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, we call $\zeta_{\mathrm{abs}}(\overline{D}) := \zeta_1(\overline{D}, X) = \inf_{x\in X(\overline{K})} \widehat{h}_{\overline{D}}(x)$ the absolute minimum of $\overline{D}$. The goal of this section is to prove the following statement, which refines Theorem 1.2 in the introduction.

Theorem 5.1. *Let $\overline{D} = (D, (g_v)_{v\in\Sigma_K})$ be a semi-positive adelic $\mathbb{R}$-Cartier divisor on X. If D is ample, there exists a subvariety $Y \subseteq X$ such that*

$$\zeta_{\mathrm{abs}}(\overline{D}) = \widehat{h}_{\overline{D}}(Y) = \min_{Z\subseteq X} \widehat{h}_{\overline{D}}(Z),$$

where the minimum is over the subvarieties $Z \subseteq X$. Moreover, $\zeta_{\mathrm{abs}}(\overline{D}) = \zeta_i(\overline{D}, X) = \zeta_i(\overline{D}, Y)$ for any $i \in \{1, \ldots, \dim Y + 1\}$.

We begin with two preliminary lemmas.

Lemma 5.2. *Let $\overline{D} = (D, (g_v)_{v\in\Sigma_K})$ be a semi-positive adelic $\mathbb{R}$-Cartier divisor on X. Assume that D is ample. Then for any subvariety $Z \subseteq X$, the following conditions are equivalent:*

(1) $\widehat{h}_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq Z$;
(2) $\zeta_1(\overline{D}, Z) > 0$.

Proof. (1) $\Longrightarrow$ (2): If $Z = \{x\}$ is a point, then $\zeta_1(\overline{D}, Z) = \widehat{h}_{\overline{D}}(x) > 0$. We assume by induction that $\dim Z > 0$ and that $\zeta_1(\overline{D}, Y) > 0$ for every subvariety $Y \subsetneq Z$. Since $\widehat{h}_{\overline{D}}(Z) > 0$, it follows from Theorem 4.3 that there exists a closed subset $Y \subsetneq Z$ such that $\inf_{x\in Z(\overline{K})\setminus Y} \widehat{h}_{\overline{D}}(x) > 0$. On the other hand, if $Y_1, \dots, Y_\ell$ are the irreducible components of Y then

$$\inf_{x\in Z(\overline{K})\cap Y} \widehat{h}_{\overline{D}}(x) = \min_{1\leq i\leq \ell} \zeta_1(\overline{D}, Y_i) > 0$$

by the induction hypothesis. Therefore we have

$$\zeta_1(\overline{D}, Z) = \min\{\inf_{x\in Z(\overline{K})\setminus Y} \widehat{h}_{\overline{D}}(x), \inf_{x\in Z(\overline{K})\cap Y} \widehat{h}_{\overline{D}}(x)\} > 0.$$

(2) $\Longrightarrow$ (1): For any subvariety $Y \subseteq Z$, we have

$$\widehat{h}_{\overline{D}}(Y) \geq \zeta_1(\overline{D}, Y) \geq \zeta_1(\overline{D}, Z) > 0,$$

where the first inequality is given by Theorem 4.3 and the second one follows from the definitions. □

Lemma 5.3. *Let $\overline{D} = (D, (g_v)_{v\in\Sigma_K})$ be a semi-positive adelic $\mathbb{R}$-Cartier divisor on X with D ample. Then*

$$\zeta_{\mathrm{abs}}(\overline{D}) = \inf_{Z\subseteq X} \widehat{h}_{\overline{D}}(Z),$$

where the infimum is over the subvarieties $Z \subseteq X$.

Proof. By Zhang's Theorem 4.3, we have

$$\widehat{h}_{\overline{D}}(Z) \geq \zeta_1(\overline{D}, Z) \geq \zeta_{\mathrm{abs}}(\overline{D})$$

for any subvariety $Z \subseteq X$, and we deduce one inequality of the lemma by taking the infimum on Z. The converse inequality follows directly from the definition of $\zeta_{\mathrm{abs}}(\overline{D})$.

□

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let $\zeta = \zeta_{\mathrm{abs}}(\overline{D}) \in \mathbb{R}$. Note that $\overline{D}(\zeta)$ is semi-positive and $\zeta_{\mathrm{abs}}(\overline{D}(\zeta)) = \zeta_{\mathrm{abs}}(\overline{D}) - \zeta = 0$. By Theorem 4.3, we have

$$\widehat{h}_{\overline{D}(\zeta)}(Y) \geq \zeta_1(\overline{D}(\zeta), Y) \geq \zeta_{\mathrm{abs}}(\overline{D}(\zeta)) = 0$$

for every subvariety $Y \subseteq X$. By Lemma 5.2 applied to $Z = X$, there exists a subvariety $Y \subseteq X$ such that $\widehat{h}_{\overline{D}(\zeta)}(Y) = 0$. Therefore Lemma 3.7 gives

$$\zeta_{\mathrm{abs}}(\overline{D}) = \widehat{h}_{\overline{D}}(Y) - \widehat{h}_{\overline{D}(\zeta)}(Y) = \widehat{h}_{\overline{D}}(Y).$$

The fact that $\zeta_{\mathrm{abs}}(\overline{D})$ coincides with the minimum in the theorem follows from Lemma 5.3. Finally, we observe that $\zeta_1(\overline{D}, Y) \geq \zeta_{\mathrm{abs}}(\overline{D}) = \widehat{h}_{\overline{D}}(Y)$ and

$$\zeta_{\dim Y+1}(\overline{D}, Y) \geq \zeta_i(\overline{D}, X) \geq \zeta_{\mathrm{abs}}(\overline{D})$$

for every $i \in \{1, \dots, \dim Y + 1\}$. Therefore Zhang's Theorem 4.3 implies that $\zeta_{\mathrm{abs}}(\overline{D}) = \zeta_i(\overline{D}, Y) = \zeta_i(\overline{D}, X)$ for every integer $1 \leq i \leq \dim Y + 1$. □

Remark 5.4. As pointed out by an anonymous referee, it is natural to ask whether the subvariety $Y \subseteq X$ of Theorem 5.1 can be chosen to be zero-dimensional in general. Equivalently, does there always exist a point $x \in X(\overline{K})$ such that $\widehat{h}_{\overline{D}}(x) = \zeta_{\mathrm{abs}}(\overline{D})$ under the assumptions of Theorem 5.1? Although it seems quite plausible to me that this question has a negative answer, I am not aware of any counterexample at the time of writing.

6. Proof of Theorem 1.1

Given an adelic $\mathbb{R}$-Cartier divisor $\overline{D}$ on X, we introduce the invariant

$$\theta(\overline{D}) := \sup\{t \in \mathbb{R} \mid \overline{D}(t) \text{ is w-ample}\} \in \mathbb{R} \cup \{-\infty\}$$

(with the convention that $\sup \emptyset = -\infty$). The main result of this section is the following theorem, from which we shall deduce Theorem 1.1 (see Corollary 6.4 below).

Theorem 6.1. *Let $\overline{D} = (D, (g_v)_{v\in\Sigma_K})$ be a semi-positive adelic $\mathbb{R}$-Cartier divisor on X. If D is ample, then $\zeta_{\mathrm{abs}}(\overline{D}) = \theta(\overline{D})$.*

Before proving this theorem, we gather some basic properties satisfied by the invariant $\theta(\overline{D})$ in the following lemma.

Lemma 6.2. *Let $\overline{D} = (D, (g_v)_{v\in\Sigma_K}) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$.*

(1) For any $\overline{D}' \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, we have

$$\theta(\overline{D} + \overline{D}') \geq \theta(\overline{D}) + \theta(\overline{D}').$$

(2) D is ample if and only if $\theta(\overline{D})$ is finite.

(3) Let $\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_\ell \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. If D is ample, then

$$\lim_{\max\{|t_1|,\ldots,|t_\ell|\}\to 0} \theta(\overline{D} + t_1\overline{D}_1 + \cdots + t_\ell\overline{D}_\ell) = \theta(\overline{D}).$$

Proof. (1) Clearly we may assume that $\theta(\overline{D}) > -\infty$ and $\theta(\overline{D}') > -\infty$. It suffices to observe that the sum of two w-ample adelic $\mathbb{R}$-Cartier divisors is w-ample.

(2) If $\theta(\overline{D})$ is finite, then clearly D is ample. Conversely, assume that D is ample. By Lemma 3.14, there exists $t \in \mathbb{R}$ such that $\overline{D}(t)$ is w-ample. Therefore $\theta(\overline{D}) \geq t$ is finite.

(3) If we replace $\overline{D}$ by $\overline{D}(t)$ for some real number t, both sides of the equality differ by $-t$. By Lemma 3.14, we may therefore assume that $\overline{D}$ is w-ample. Let $\varepsilon > 0$ be a real number. For sufficiently small real numbers $t_1, \ldots, t_\ell$, the adelic $\mathbb{R}$-Cartier divisors

$$(1+\varepsilon)\overline{D} - (\overline{D} + t_1\overline{D}_1 + \cdots t_\ell\overline{D_\ell}) = \varepsilon\overline{D} - (t_1\overline{D}_1 + \cdots + t_\ell\overline{D_\ell})$$

and

$$\overline{D} + t_1\overline{D}_1 + \cdots + t_\ell\overline{D_\ell} - (1-\varepsilon)\overline{D} = \varepsilon\overline{D} + (t_1\overline{D}_1 + \cdots + t_\ell\overline{D_\ell})$$

are w-ample by Lemma 3.11. In particular,

$$\theta(\varepsilon\overline{D} - (t_1\overline{D}_1 + \cdots + t_\ell\overline{D_\ell})) \geq 0 \quad \text{and} \quad \theta(\varepsilon\overline{D} + (t_1\overline{D}_1 + \cdots + t_\ell\overline{D_\ell})) \geq 0$$

by definition of θ. By (1), we infer that

$$(1+\varepsilon)\theta(\overline{D}) \geq \theta(\overline{D} + t_1\overline{D}_1 + \cdots + t_\ell\overline{D_\ell}) \geq (1-\varepsilon)\theta(\overline{D}),$$

and the result follows. □

Let us now prove Theorem 6.1. We shall combine Zhang's arithmetic Nakai-Moishezon criterion [Zha95a, Theorem 4.2] and the continuity property given by Lemma 6.2 (3).

Proof of Theorem 6.1. Since D is ample, we have $\theta(\overline{D}) > -\infty$ by Lemma 6.2 (2). Let $t < \theta(\overline{D})$ be a real number. By definition, $\overline{D}(t)$ is w-ample and Lemma 3.13 gives

$$\zeta_{\mathrm{abs}}(\overline{D}) - t = \zeta_{\mathrm{abs}}(\overline{D}(t)) > 0.$$

By letting t tend to $\theta(\overline{D})$, we conclude that $\zeta_{\mathrm{abs}}(\overline{D}) \geq \theta(\overline{D})$.

For the converse inequality, let us first assume that $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{Q}}$. By homogeneity of $\theta(\overline{D})$ and $\zeta_{\mathrm{abs}}(\overline{D})$, we may assume that $\overline{D}$ is an adelic Cartier divisor without loss of generality. Let $t < \zeta_{\rm abs}(\overline{D})$ be a real number. Since $\zeta_{\rm abs}(\overline{D}(t)) = \zeta_{\rm abs}(\overline{D}) - t > 0$, we have

$$h_{\overline{D}(t)}(Y) > 0$$

for any subvariety $Y \subseteq X$ by Lemma 5.2. By the arithmetic Hilbert-Samuel formula [Zha95b, Theorem 1.7] (see also [Zha95b, Proof of Theorem 1.8]), for any subvariety $Y \subseteq X$ there exists an integer $n > 0$ such that $\widehat{H}^0(Y, n\overline{D}(t)_{|Y}) \neq 0$. By [CM18, Theorem 1.2], $\overline{D}(t)$ is w-ample. Therefore $\theta(\overline{D}) \geq t$, and we conclude by letting t tend to $\zeta_{\rm abs}(\overline{D})$.

Let us now prove the equality $\zeta_{\rm abs}(\overline{D}) = \theta(\overline{D})$ in full generality. Since D is ample, we can write $D = \sum_{i=1}^{\ell} \lambda_i A_i$ where for each $i \in \{1, \dots, \ell\}$, $\lambda_i \in \mathbb{R}_{>0}$ and A_i is an ample Cartier divisor on X. For each $i \in \{1, \dots, \ell\}$, we equip A_i with a collection of A_i-Green functions $(g_{i,v})_{v\in\Sigma_K}$ such that $\overline{A_i} = (A_i, (g_{i,v})_{v\in\Sigma_K}) \in \widehat{\rm Div}(X)$ is semi-positive. For any $\varepsilon > 0$, we can find a ℓ-tuple of real numbers $\mathbf{t} = (t_1, \dots, t_\ell) \in [0, \varepsilon]^\ell$ such that

$$\overline{D}_{\mathbf{t}} := \overline{D} + \sum_{i=1}^{\ell} t_i \overline{A}_i = (\sum_{i=1}^{\ell} (\lambda_i + t_i) A_i, (g_v + \sum_{i=1}^{\ell} t_i g_{i,v})_{v\in\Sigma_K}) \in \widehat{\rm Div}(X)_{\mathbb{Q}}$$

is an adelic $\mathbb{Q}$-Cartier divisor. Note that $\overline{D}_{\mathbf{t}}$ is semi-positive. By the above, we have $\zeta_{\rm abs}(\overline{D}_{\mathbf{t}}) = \theta(\overline{D}_{\mathbf{t}})$. Letting ε tend to zero, we find that $\zeta_{\rm abs}(\overline{D}) = \theta(\overline{D})$ by continuity of $\zeta_{\rm abs}$ and θ (Lemma 4.1 (2) and Lemma 6.2 (3)). □

Remark 6.3. In the proof of Theorem 6.1, we used a particular case of a theorem of Chen and Moriwaki [CM18], which generalizes Zhang's arithmetic Nakai-Moishezon criterion [Zha95a, Theorem 4.2]. Using Zhang's original result would have required extra work since it involves stronger assumptions on the metrics.

We now deduce a refinement of Theorem 1.1 from Theorems 5.1 and 6.1.

Corollary 6.4. *Let $\overline{D} = (D, (g_v)_{v\in\Sigma_K})$ be a semi-positive adelic $\mathbb{R}$-Cartier divisor on X. The following conditions are equivalent:*

(1) $\overline{D}$ is ample;
(2) $h_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq X$;
(3) D is ample and $\inf_{Y\subseteq X} \widehat{h}_{\overline{D}}(Y) > 0$, where the infimum is over all subvarieties $Y \subseteq X$;
(4) D is ample and $\zeta_{\rm abs}(\overline{D}) > 0$.

Proof. The assertion (2) ⇔ (3) ⇔ (4) is given by Lemma 3.9 and Theorem 5.1. The implication (1) ⇒ (4) is Lemma 3.13, so it only remains to prove (4) ⇒ (1). If (4) holds, then $\theta(\overline{D}) = \zeta_{\rm abs}(\overline{D}) > 0$ by Theorem 6.1 and therefore $\overline{D}$ is w-ample by definition of $\theta(\overline{D})$. Since $\overline{D}$ is also semi-positive, it is ample. □

Remark 6.5. In [BGMPS16, Definition 3.18 (2)], the authors defined arithmetic ampleness by using the notion of metrized divisors generated by small $\mathbb{R}$-sections. It is straightforward to check that if $\overline{D} \in \widehat{\rm Div}(X)_{\mathbb{R}}$ is ample in the sense of Definition 3.10, then it is ample in the sense of [BGMPS16]. On the other hand, if $\overline{D}$ is ample in the sense of [BGMPS16], then clearly $\zeta_{\rm abs}(\overline{D}) > 0$. Therefore, Corollary 6.4 implies that our definition of arithmetic ampleness coincides with the one of [BGMPS16, Definition 3.18 (2)].

We conclude this article with two direct consequences of our results.

Corollary 6.6. *Let $\overline{D} = (D, (g_v)_{v\in\Sigma_K}) \in \widehat{\rm Div}(X)_{\mathbb{R}}$ be semi-positive and let $\overline{A} \in \widehat{\rm Div}(X)_{\mathbb{R}}$ be w-ample. The following assertions are equivalent:*

(1) $\overline{D}$ is ample;

(2) D is ample and there exists a real number $\varepsilon > 0$ such that $\widehat{h}_{\overline{D}}(x) \geq \varepsilon \widehat{h}_{\overline{A}}(x)$ for any $x \in X(\overline{K})$.

Proof. (1) $\Longrightarrow$ (2): By Lemma 3.11, there exists a real number $\varepsilon > 0$ such that $\overline{D} - \varepsilon\overline{A}$ is w-ample. By Lemma 3.13, we have

$$\widehat{h}_{\overline{D}}(x) - \varepsilon \widehat{h}_{\overline{A}}(x) = \widehat{h}_{\overline{D}-\varepsilon\overline{A}}(x) > 0$$

for any $x \in X(\overline{K})$.

(2) $\Longrightarrow$ (1): Since $\overline{A}$ is w-ample, $\zeta_{\mathrm{abs}}(\overline{A}) > 0$ by Lemma 3.13. Assumption (2) therefore implies that $\zeta_{\mathrm{abs}}(\overline{D} - \varepsilon'\overline{A}) > 0$ for any $\varepsilon' \in (0, \varepsilon)$. By Lemma 4.1 (1), it follows that

$$\zeta_{\mathrm{abs}}(\overline{D}) \geq \zeta_{\mathrm{abs}}(\overline{D} - \varepsilon'\overline{A}) + \zeta_{\mathrm{abs}}(\varepsilon'\overline{A}) > 0,$$

and therefore $\overline{D}$ is ample by Corollary 6.4. □

Corollary 6.7. *Let $\overline{D} = (D, (g_v)_{v\in\Sigma_K}) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be semi-positive. The following assertions are equivalent:*

(1) $\zeta_{\mathrm{abs}}(\overline{D}) \geq 0$;

(2) $\overline{D} + \overline{A}$ is ample for any ample $\overline{A} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$.

Proof. (1) $\Longrightarrow$ (2): Let $\overline{A} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be ample. Then the underlying divisor A of $\overline{A}$ is ample. Since D is nef by semi-positivity of $\overline{D}$, $D + A$ is ample. Moreover we have

$$\zeta_{\mathrm{abs}}(\overline{D} + \overline{A}) \geq \zeta_{\mathrm{abs}}(\overline{D}) + \zeta_{\mathrm{abs}}(\overline{A}) \geq \zeta_{\mathrm{abs}}(\overline{A}) > 0,$$

where the last inequality is given by Lemma 3.13. By Corollary 6.4, $\overline{D}+\overline{A}$ is ample.

(2) $\Longrightarrow$ (1): Let $x \in X(\overline{K})$ be a closed point. We want to prove that $\widehat{h}_{\overline{D}}(x) \geq 0$. Let $\overline{A} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be ample and semi-positive and let $\varepsilon > 0$ be a real number. Since $\overline{D} + \varepsilon\overline{A}$ is ample, we have

$$\widehat{h}_{\overline{D}}(x) + \varepsilon \widehat{h}_{\overline{A}}(x) = \widehat{h}_{\overline{D}+\varepsilon\overline{A}}(x) > 0,$$

and we conclude by letting ε tend to zero. □

A semi-positive adelic ℝ-Cartier divisor satisfying $\zeta_{\mathrm{abs}}(\overline{D}) \geq 0$ is usually called nef in the literature [Mor16, Definition 4.4.1]. Roughly speaking, Corollary 6.7 asserts that an adelic ℝ-Cartier divisor is nef if and only if it is the limit of a sequence of ample adelic ℝ-Cartier divisors.

Acknowledgements

I thank the anonymous referees for valuable comments and suggestions.

References

[BG06] Enrico Bombieri and Walter Gubler. *Heights in Diophantine geometry*, volume 4 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2006.

[BGMPS16] José Ignacio Burgos Gil, Atsushi Moriwaki, Patrice Philippon, and Martín Sombra. Arithmetic positivity on toric varieties. *J. Algebraic Geom.*, 25(2):201–272, 2016.

[BGPS14] José Ignacio Burgos Gil, Patrice Philippon, and Martín Sombra. Arithmetic geometry of toric varieties. Metrics, measures and heights. *Astérisque*, (360):vi+222, 2014.

[CM18] Huayi Chen and Atsushi Moriwaki. Extension property of semipositive invertible sheaves over a non-archimedean field. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 18(1):241–282, 2018.

[CP90] Frédéric Campana and Thomas Peternell. Algebraicity of the ample cone of projective varieties. *J. Reine Angew. Math.*, 407:160–166, 1990.

[Iko16] Hideaki Ikoma. Remarks on arithmetic restricted volumes and arithmetic base loci. *Publ. Res. Inst. Math. Sci.*, 52(4):435–495, 2016.

[Iko21] Hideaki Ikoma. Adelic Cartier divisors with base conditions and the Bonnesen–Diskant-type inequalities. *Tohoku Mathematical Journal*, 73(3):341 – 401, 2021.

[Mor15] Atsushi Moriwaki. Semiample invertible sheaves with semipositive continuous hermitian metrics. *Algebra Number Theory*, 9(2):503–509, 2015.

[Mor16] Atsushi Moriwaki. Adelic divisors on arithmetic varieties. *Mem. Amer. Math. Soc.*, 242(1144):v+122, 2016.

[Zha95a] Shouwu Zhang. Positive line bundles on arithmetic varieties. *J. Amer. Math. Soc.*, 8(1):187–221, 1995.

[Zha95b] Shouwu Zhang. Small points and adelic metrics. *J. Algebraic Geom.*, 4(2):281–300, 1995.