APPROXIMATION OF ADELIC DIVISORS AND EQUIDISTRIBUTION OF SMALL POINTS

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Abstract. We study the asymptotic distribution of the Galois orbits of generic sequences of algebraic points of small height in a projective variety over a number field. Our main result is a generalization of Yuan's equidistribution theorem that applies to heights for which Zhang's lower bound for the essential minimum is not necessarily an equality. It extends to all projective varieties a theorem of Burgos Gil, Philippon, Rivera-Letelier and the second author for toric varieties. It also applies to sums of canonical heights for an algebraic dynamical system, and in particular it recovers Kühne's semiabelian equidistribution theorem. We also generalize previous work of Chambert-Loir and Thuillier to obtain new logarithmic equidistribution results. Finally we extend our main result to the quasi-projective setting recently introduced by Yuan and Zhang.

Contents

Introduction

In their seminal work [SUZ97], Szpiro, Ullmo and Zhang used Arakelov theory to prove an equidistribution theorem for the Galois orbits of algebraic points of a projective variety over a number field. Their result applies to generic sequences of points in an abelian variety with Néron-Tate heights converging to zero, and is at the heart of the proof of the Bogomolov conjecture for abelian varieties [Ull98, Zha98]. It has been developed in several directions by many authors, culminating with the celebrated equidistribution theorem of Yuan [Yua08].

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In this paper we generalize Yuan's theorem to allow more flexibility in the choice of the height function. In the same spirit we generalize the logarithmic equidistribution theorem of Chambert-Loir and Thuillier [CT09].

Our results extend to all projective varieties the toric equidistribution theorem of Burgos Gil, Philippon, Rivera-Letelier and the second author [BPRS19]. Moreover they strengthen it, showing that this result holds without any semipositivity assumption and with respect to functions admitting logarithmic singularities along some specific effective divisors. They also apply in the dynamical setting to sums of canonical height functions, giving an equidistribution theorem allowing test functions with logarithmic singularities along preperiodic hypersurfaces. In particular, this recovers the semiabelian equidistribution theorem of Kühne [Küh22] and strengthens its statement to include convergence with respect to functions with logarithmic singularities along torsion and boundary hypersurfaces.

We also give a partial converse to our main result, showing that in some situations the imposed condition on the height function is necessary for the the equidistribution to occur. This includes the semipositive toric case, thus recovering the reciprocal of the toric equidistribution theorem from [BPRS19]. Finally we extend our main result to the setting of adelic line bundles on quasi-projective varieties introduced by Yuan and Zhang [YZ26].

Background. Arakelov geometry provides a very general and powerful framework to define and study heights of algebraic points. Classical heights in Diophantine geometry such as Néron-Tate heights on abelian varieties are special cases of height functions associated to metrized line bundles in the sense of Zhang [Zha95b].

Let K be a number field with a fixed algebraic closure $\overline{K}$. Let X be a projective variety of dimension $d \geq 1$ over K and $\overline{D}$ an adelic divisor on X. The latter consists of a Cartier divisor D on X with an adelic family of Green functions, and is a datum essentially equivalent to that of a metrized line bundle on X. Let $h_{\overline{D}} \colon X(\overline{K}) \to \mathbb{R}$ be the associated height function, and denote by $\mu^{\mathrm{abs}}(\overline{D})$ and $\mu^{\mathrm{ess}}(\overline{D})$ its absolute and essential minima.

A fundamental inequality of Zhang [Zha95a] asserts that if D is ample and $\overline{D}$ is semipositive then

$$\mu^{\mathrm{ess}}(\overline{D}) \geq \frac{(\overline{D}^{d+1})}{(d+1)(D^d)}, \tag{1}$$

where (D^d) and $(\overline{D}^{d+1})$ denote the top intersection numbers of D and $\overline{D}$, respectively.

For every generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ we have $\liminf_{\ell\to\infty} h_{\overline{D}}(x_\ell) \geq \mu^{\mathrm{ess}}(\overline{D})$, and there exist generic sequences for which the equality holds. Following [BPRS19], we say that $(x_\ell)_\ell$ is $\overline{D}$-*small* if the heights of these points converge to the smallest possible value, that is

$$\lim_{\ell\to\infty} h_{\overline{D}}(x_\ell) = \mu^{\mathrm{ess}}(\overline{D}).$$

For each place $v \in \mathfrak{M}_K$ we denote by X_v^{an} the v-adic analytification of X. It is a Berkovich space over $\mathbb{C}_v$, the completion of an algebraic closure of the local field K_v. For an algebraic point $x \in X(\overline{K})$ we denote by $\delta_{O(x)_v}$ the uniform probability measure on $O(x)_v \subset X_v^{\mathrm{an}}$, the image in X_v^{an} of the Galois orbit of x.

With this notation, Yuan's theorem [Yua08] can be stated as follows.

Theorem 1 (Yuan). *Let $\overline{D} \in \widehat{\mathrm{Div}}(X)$ be a semipositive adelic divisor on X with ample geometric divisor D, and assume that*

$$\mu^{\mathrm{ess}}(\overline{D}) = \frac{(\overline{D}^{d+1})}{(d+1)(D^d)}. \tag{2}$$

Then for every $v \in \mathfrak{M}_K$ and every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ the sequence of probability measures $(\delta_{O(x_\ell)_v})_\ell$ on X_v^{an} converges weakly to $c_1(\overline{D}_v)^{\wedge d}/(D^d)$.

In other words, this result asserts that if Zhang's lower bound (1) is an equality then $\overline{D}$ has the equidistribution property at every place $v \in \mathfrak{M}_K$, in the sense that the Galois orbits of points in $\overline{D}$-small generic sequences equidistribute in X_v^{an} for every v, and that moreover the v-adic equidistribution measure is the normalized v-adic Monge-Ampère measure of $\overline{D}$. When X is a curve, this theorem is due to Autissier [Aut01] and Chambert-Loir [Cha06].

Yuan's theorem encompasses in a unified way the (Archimedean) theorems of Szpiro, Ullmo and Zhang for abelian varieties [SUZ97], of Bilu for canonical heights on toric varieties [Bil97] and of Chambert-Loir for canonical heights on isotrivial semiabelian varieties [Cha00], as well as their non-Archimedean analogues by Chambert-Loir [Cha06]. The positivity assumptions in Theorem 1 can be weakened, and in fact Yuan's proof remains valid when D is big but not necessarily ample. Moreover, Berman and Boucksom [BB10] and later Chen [Che11] generalized this theorem for the Archimedean places to the non-semipositive case.

By a result of the first author, the fact that Zhang's lower bound is an equality is equivalent to the equality between the essential and the absolute minima [Bal24, Theorem 6.6]. This is a very restrictive condition that is nevertheless satisfied in the important case of canonical heights on polarized dynamical systems [Zha95b], which includes the canonical heights on toric varieties and the Néron-Tate heights on abelian varieties.

To our knowledge, there is no general result ensuring the equidistribution property for an adelic divisor on a projective variety over a number field when Zhang's inequality is strict. However, there are two remarkable situations where results in this direction are known.

In [BPRS19], Burgos Gil, Philippon, Rivera-Letelier and the second author achieved a systematic description of this property in the toric setting. Their result gives a criterion in terms of convex analysis, and shows that there are plenty of toric adelic divisors for which Zhang's inequality is strict but that nevertheless satisfy the property. This provides a wealth of new equidistribution phenomena previously out of reach, as well as situations where it does not occur.

In [Küh22], Kühne proved the long standing semiabelian equidistribution conjecture, showing that this property holds for canonical heights on semiabelian varieties. As shown by Chambert-Loir [Cha00], Theorem 1 does not apply in this case as the condition (2) can fail when the semiabelian variety is not isotrivial. Kühne's theorem allowed him to give a purely Arakelov-geometric proof of the semiabelian Bogomolov conjecture, previously established by David and Philippon with other methods [DP00].

Main theorem. The results of [BPRS19] and [Küh22] raise the following question: on an arbitrary projective variety, what can be said regarding the equidistribution property for an adelic divisor when Zhang's lower bound is strict? More precisely, can we identify a condition weaker than (2) that guarantees this property for a given adelic divisor? Our main contribution is a generalization of Yuan's theorem that answers affirmatively this question.

As in [SUZ97, Cha06, Yua08, Che11], the positivity properties of adelic divisors play a central role in our approach. We work with the more general notion of adelic $\mathbb{R}$-divisors developed by Moriwaki, as it provides a particularly efficient framework to study positivity [BMPS16, Mor16]. Adelic $\mathbb{R}$-divisors are better behaved on normal varieties, and so we assume that X is normal from now on.

Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor on X. A *semipositive approximation* of $\overline{D}$ is a pair $(\phi, \overline{Q})$ consisting of a normal modification $\phi\colon X' \to X$ and a semipositive adelic $\mathbb{R}$-divisor $\overline{Q}$ on X' such that the $\mathbb{R}$-divisor Q is big and $\phi^*\overline{D} - \overline{Q}$ is pseudo-effective. It is a variant of the notion of admissible decomposition introduced by Chen [Che11].

Given two big $\mathbb{R}$-divisors P, A on X, the *inradius* of P with respect to A is the positive real number defined as

$$r(P; A) = \sup\{\lambda \in \mathbb{R} \mid P - \lambda A \text{ is big}\}.$$

This geometric invariant was introduced by Teissier [Tei82] and measures the bigness of P, see Section 1.2 for more details. Our main result is the following.

Theorem 2. *Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor on X with D big. Assume that there exists a sequence $(\phi_n\colon X_n \to X, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ such that*

$$\lim_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; \phi_n^* D)} = 0. \tag{3}$$

Let $v \in \mathfrak{M}_K$, and for each $n \geq 1$ let $\nu_{n,v}$ be the pushforward to X_v^{an} of the normalized v-adic Monge-Ampère measure $c_1(\overline{Q}_{n,v})^{\wedge d}/(Q_n^d)$ on $X_{n,v}^{\mathrm{an}}$. Then

(i) the sequence $(\nu_{n,v})_n$ converges weakly to a probability measure ν_v on X_v^{an},

(ii) for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$, the sequence of probability measures $(\delta_{O(x_\ell)_v})_\ell$ on X_v^{an} converges weakly to ν_v.

Theorem 1 follows immediately from this result applied with the constant sequence $(\phi_n, \overline{Q}_n) = (\mathrm{Id}_X, \overline{D})$, $n \in \mathbb{N}$. Theorem 2 also implies Chen's equidistribution theorem (Corollary 4.12).

We actually prove a stronger result (Theorem 4.8) showing that under the condition (3) for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ and $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ we have

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \lim_{n\to\infty} \frac{(\overline{Q}_n^d \cdot \phi_n^*\overline{E}) - d\,\mu^{\mathrm{ess}}(\overline{D})\,(Q_n^{d-1} \cdot \phi_n^* E)}{(Q_n^d)}. \tag{4}$$

In particular, both limits exist in $\mathbb{R}$ and the second does not depend on the choice of the sequence $(\phi_n, \overline{Q}_n)_n$. Theorem 2 follows by specializing (4) to the adelic divisors $\overline{E}$ over the zero divisor of X associated to continuous real-valued functions on X_v^{an}.

We also obtain a generalization of Chambert-Loir and Thuillier's logarithmic equidistribution theorem [CT09], showing that in the situation of Theorem 2 the equidistribution property extends to test functions with logarithmic singularities along effective divisors satisfying a numerical condition (Theorem 5.13 and Corollary 5.15).

Note that it is always possible to construct a sequence $(\phi_n, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ with $\mu^{\mathrm{abs}}(\overline{Q}_n)$ converging to $\mu^{\mathrm{ess}}(\overline{D})$, see for instance Lemma 4.13. However, for such sequences the condition (3) is not necessarily satisfied since as already explained, there are situations where the equidistribution property fails.

Toric varieties. We first apply our results in the toric setting. To this end, let X be a projective toric variety over K with torus $\mathbb{T} \simeq \mathbb{G}_{\mathrm{m}}^d$ and $\overline{D}$ a toric adelic $\mathbb{R}$-divisor on X with big geometric $\mathbb{R}$-divisor D. Let Δ_D be the d-dimensional polytope associated to this $\mathbb{R}$-divisor. Following [BPS14], the family of Green functions of $\overline{D}$ induces a family of concave functions $\vartheta_{\overline{D},v}\colon \Delta_D \to \mathbb{R}$, $v \in \mathfrak{M}_K$, called *local roof functions*, whose weighted sum gives the *global roof function*

$$\vartheta_{\overline{D}}\colon \Delta_D \to \mathbb{R}.$$

These concave functions convey a lot of information about the height function of $\overline{D}$. For instance, its essential minimum coincides with the maximum of $\vartheta_{\overline{D}}$, and if $\overline{D}$ is semipositive then its absolute minimum coincides with the minimum of this concave function. This readily implies that the only toric adelic $\mathbb{R}$-divisors to which Yuan's theorem applies are those whose associated global roof function is constant.

The global roof function $\vartheta_{\overline{D}}$ is said to be *wide* if the width of its sup-level sets remains relatively large as the level approaches its maximum, see Appendix A for details. When this is the case, there is a unique balanced family of vectors u_v, $v \in \mathfrak{M}_K$, such that each u_v is a sup-gradient of the v-adic roof function $\vartheta_{\overline{D},v}$. Then for each v we can associate to u_v a probability measure on X_v^{an} that we denote by $\eta_{\overline{D},v}$. When v is Archimedean, it is the Haar probability measure on a translate of the compact torus $\mathbb{S}_v \simeq (S^1)^d$ of $\mathbb{T}_v^{\mathrm{an}}$, whereas if v is non-Archimedean it is the Dirac measure at a translate of the Gauss point of this v-adic analytic torus, see Section 6.2 for precisions.

The following is our main result in this setting.

Theorem 3. *Let $\overline{D}$ be a toric adelic $\mathbb{R}$-divisor on X with D big, and assume that $\vartheta_{\overline{D}}$ is wide. Then for every $v \in \mathfrak{M}_K$ and every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ the sequence of probability measures $(\delta_{O(x_\ell)_v})_\ell$ on X_v^{an} converges weakly to $\eta_{\overline{D},v}$.*

We actually show a stronger result (Theorem 6.6): under the assumptions of Theorem 3, for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ and every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with E toric we have

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\eta_{\overline{D},v},$$

where $g_{\overline{E},v}$ denotes the v-adic Green function of $\overline{E}$. The case when E is arbitrary can be reduced to the previous one by linear equivalence (Corollary 6.8).

As an application of our logarithmic equidistribution result we strengthen the previous to allow test functions with logarithmic singularities along the boundary and some specific translates of subtori (Theorem 6.12). In the semipositive case we can combine it with the characterization of the Bogomolov property in [BPRS19, Section 5] to obtain the following consequence (Corollary 6.13).

Theorem 4. *Let $\overline{D}$ be a semipositive toric adelic $\mathbb{R}$-divisor on X with D big and such that $\vartheta_{\overline{D}}$ is wide. Let E be an effective divisor on X such that each irreducible component V of E that is not contained in $X \setminus \mathbb{T}$ satisfies $\mu^{\mathrm{ess}}(\overline{D}|_V) = \mu^{\mathrm{ess}}(\overline{D})$. Then for every $v \in \mathfrak{M}_K$ and every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ we have*

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \varphi\, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \varphi\, d\eta_{\overline{D},v}$$

for any function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R}\cup\{\pm\infty\}$ with at most logarithmic singularities along E.

We also show that when $\overline{D}$ is semipositive, the converse of Theorem 3 holds: in this situation, the condition that $\vartheta_{\overline{D}}$ is wide is necessary for the equidistribution of the Galois orbits of points in $\overline{D}$-small generic sequences (Theorem 6.9). Together with Theorem 3, this fully recover the main theorem of [BPRS19]. We refer to Remark 6.11 for a more detailed comparison between our results and those in *loc. cit.*.

Dynamical systems and semiabelian varieties. Yuan's theorem gives the equidistribution property for the canonical metrized line bundles associated to polarized dynamical systems, a result with vast consequences in arithmetic dynamics. In a similar vein, our result implies this property for adelic $\mathbb{R}$-divisors that are sums of several canonical adelic $\mathbb{R}$-divisors with different regimes with respect to a dynamical system that is not necessarily polarized.

Let $\phi\colon X \to X$ be a surjective endomorphism of a normal projective variety over K of dimension $d \geq 1$. Then ϕ is finite and we denote by $\deg(\phi)$ its degree. For $i = 1, \dots, s$ let $D_i \in \mathrm{Div}(X)_{\mathbb{R}}$ with $\phi^* D_i \equiv q_i D_i$ for a real number $q_i > 1$ and set

$$\overline{D} = \sum_{i=1}^{s} \overline{D}_i^{\mathrm{can}},$$

where $\overline{D}_i^{\mathrm{can}} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ denotes the canonical adelic $\mathbb{R}$-divisor over D_i. For simplicity here we assume that D_i is effective for every i and that D is ample, although our results are valid under slightly weaker positivity assumptions.

Theorem 5. *Let $v \in \mathfrak{M}_K$ and denote by μ_v the normalized Monge-Ampère measure of any semipositive adelic $\mathbb{R}$-divisor over D. Then*

(i) the sequence $\left(\dfrac{\phi_v^{\circ n,\mathrm{an},*}\mu_v}{\deg(\phi)^n}\right)_n$ converges weakly to a probability measure ν_v on X_v^{an},

(ii) for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ the sequence of probability measures $(\delta_{O(x_\ell)_v})_\ell$ on X_v^{an} converges weakly to ν_v.

In this statement we denote by $\phi_v^{\circ n,\mathrm{an},*}\mu_v$ the pullback of the probability measure μ_v on X_v^{an} with respect to the n-th iterate of the v-adic analytification of ϕ, which is well-defined because ϕ is finite. Note that a semipositive adelic $\mathbb{R}$-divisor over D always exists by ampleness.

The probability measure ν_v in Theorem 5 is called the v-adic *equilibrium measure* of ϕ with respect to D, and this result shows that it is well-defined in this context. If the v-adic Green function of $\overline{D}_i^{\mathrm{can}}$ is semipositive for every i then this measure coincides with the normalized v-adic Monge-Ampère measure of $\overline{D}$, that is

$$\nu_v = \frac{c_1(\overline{D}_v)^{\wedge d}}{(D^d)}.$$

These results are contained in Theorem 7.4, which also shows the existence and gives an explicit expression for the limit height $\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell)$ for any $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$.

Theorem 5 is a straightforward consequence of Theorem 2. When $\overline{D}$ is semipositive, it is obtained by considering the sequence of adelic $\mathbb{R}$-divisors on X defined as

$$\overline{Q}_n = q^{-n}\, \phi^{\circ n,*}\overline{D}, \quad n \in \mathbb{N},$$

with $q = \max_j q_j$. From the dynamical properties of $\overline{D}_i^{\mathrm{can}}$, $i = 1, \dots, s$, one can check that $(\mathrm{Id}_X, \overline{Q}_n)$ is a semipositive approximation of $\overline{D}$ whose absolute minimum decreases much faster than its inradius as $n \to \infty$, and conclude then with Theorem 2.

As another application of the logarithmic equidistribution theorem, we strengthen the previous result in the semipositive case to allow test functions with logarithmic singularities along preperiodic hypersurfaces.

Theorem 6. *Assume that $\overline{D}_i^{\mathrm{can}}$ is semipositive for every i. Let $(x_\ell)_\ell$ be a $\overline{D}$-small generic sequence in $X(\overline{K})$ and E an effective divisor on X such that each of its irreducible components is preperiodic. Then for every $v \in \mathfrak{M}_K$ we have*

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \varphi \, \frac{c_1(\overline{D}_v)^{\wedge d}}{(D^d)}$$

for any function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R}\cup\{\pm\infty\}$ with at most logarithmic singularities along E.

As a particular case of these results we recover Kühne's semiabelian equidistribution theorem [Küh22] and extend it to include the computation of the corresponding limit heights (Theorem 7.10). Furthermore, we can also strengthen it to show the convergence of the Galois orbits of points in generic small sequences with respect to test functions with logarithmic singularities along torsion hypersurfaces (Theorem 7.14).

Further results and questions. In Theorem 4.16 we give an alternative formulation of Theorem 2 in terms of the arithmetic positive intersection numbers introduced by Chen [Che11]. Its statement is more intrinsic in the sense that it does not rely on the choice of a specific sequence of semipositive approximations.

In [YZ26, Theorem 5.4.3], Yuan and Zhang extended Yuan's theorem to the setting of adelic line bundles on quasi-projective varieties. In Section 8 we present a generalization of Theorem 2 in this context that recovers this result.

In the toric setting the condition in Theorem 2 translates into a convex analysis statement involving the global roof function. We have a similar result on an arbitrary projective variety in terms of the arithmetic Okounkov bodies introduced and studied by Boucksom and Chen [BC11]. Since this is beyond the scope of this text, it will appear in a subsequent manuscript.

As already explained, the condition in Theorem 2 is optimal in the semipositive toric case: for a semipositive toric adelic $\mathbb{R}$-divisor the equidistribution property at every place is equivalent to the existence of a sequence of semipositive approximations satisfying the condition (3). It is natural to ask whether this remains true in general, that is if Theorem 2 actually gives a criterion for the equidistribution property (Question 4.19). In Proposition 5.17 we give an affirmative answer under an additional technical condition, which is always satisfied for semipositive toric adelic $\mathbb{R}$-divisors.

Comments on the proof. Our initial motivation was to generalize the toric equidistribution theorem from [BPRS19] to all projective varieties. A major obstacle was that both the statement and the proof of this result rely heavily on notions and tools that are specific to toric adelic $\mathbb{R}$-divisors, with no clear extension outside of the toric setting. A first step was to produce a non-trivial reformulation of the toric equidistribution theorem that we could translate in terms of the arithmetic and geometric properties of the algebra of global sections of the adelic $\mathbb{R}$-divisor, leading to the statement of Theorem 2.

The proof of Theorem 2 is based on Szpiro, Ullmo and Zhang's variational principle and Yuan's arithmetic analogue of Siu's inequality. Compared with [Yua08], we do not apply the latter directly to $\overline{D}$ but rather to the sequence of semipositive approximations satisfying the condition (3). The main difficulty is that we need to keep a very precise control of the error terms that arise in this asymptotic process. The core of our proof is a new consequence of the arithmetic Siu's inequality with an error term involving an inradius (Corollary 5.3). It is based on precise estimates for arithmetic intersection numbers based on the arithmetic Hodge index theorem of Yuan and Zhang [YZ17]. This technical feature is not needed in the case of curves, for which most of the difficulties disappear.

We next outline the proof in this situation. Assume that X is a smooth projective curve over K with an adelic $\mathbb{R}$-divisor $\overline{D}$ such that D is big. Shifting the Green functions of $\overline{D}$ if necessary we assume $\mu^{\mathrm{ess}}(\overline{D}) > 0$.

The assumption of Theorem 2 boils down to

$$\lim_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{(Q_n)} = 0 \tag{5}$$

for a sequence $(\overline{Q}_n)_n$ of semipositive adelic $\mathbb{R}$-divisors on X with Q_n big and $\overline{D} - \overline{Q}_n$ pseudo-effective for every n. Indeed, since $d = 1$ every modification of X is an isomorphism, and moreover $r(Q_n; D) = (Q_n)/(D)$ and so the inradius in that condition can be replaced by the intersection number (Q_n).

Let $v \in \mathfrak{M}_K$ and $(x_\ell)_\ell$ a $\overline{D}$-small generic sequence in $X(\overline{K})$. To prove that $\overline{D}$ satisfies the equidistribution property at v we need to show that for every continuous function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R}$ we have

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \lim_{n\to\infty} \int_{X_v^{\mathrm{an}}} \varphi \, \frac{c_1(\overline{Q}_{n,v})^{\wedge d}}{(Q_n)}, \tag{6}$$

including the existence of both limits. By a standard density argument, it suffices to consider the case where φ is $\mathrm{Gal}(\overline{K}_v/K_v)$-invariant. This allows to associate to φ an adelic divisor $\overline{E} = \overline{0}^{\varphi}$ over the zero divisor of X such that (6) translates into

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \lim_{n\to\infty} \frac{(\overline{Q}_n \cdot \overline{E})}{(Q_n)}. \tag{7}$$

By density, we can furthermore assume that $\overline{E}$ is the difference of two semipositive adelic divisors on X.

Let $\lambda \in (0, 1)$. Then

$$\mu^{\mathrm{ess}}(\overline{D}) + \lambda \liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \liminf_{\ell\to\infty} h_{\overline{D}+\lambda\overline{E}}(x_\ell) \ge \mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E})$$

by the linearity of heights and the definition of the essential minimum. Now the assumption that $\overline{D} - \overline{Q}_n$ is pseudo-effective together with Zhang's lower bound for the essential minimum in terms of the χ-volume (Theorem 2.18) gives

$$\mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E}) \ge \mu^{\mathrm{ess}}(\overline{Q}_n + \lambda\overline{E}) \ge \frac{\widehat{\mathrm{vol}}_\chi(\overline{Q}_n + \lambda\overline{E})}{2\,(Q_n)}.$$

The condition (5) implies $\mu^{\mathrm{abs}}(\overline{Q}_n) > 0$ for n large enough, and therefore $\overline{Q}_n$ is nef. Then a classical consequence of Yuan's arithmetic version of Siu's inequality [Yua08] shows that there exists a constant $c \ge 0$ such that

$$\widehat{\mathrm{vol}}_\chi(\overline{Q}_n + \lambda\overline{E}) \ge (\overline{Q}_n^2) + 2\,\lambda\,(\overline{Q}_n \cdot \overline{E}) - c\,\lambda^2 \tag{8}$$

for every sufficiently large n. On the other hand, Zhang's theorem on successive minima gives $(\overline{Q}_n^2) \geq 2\,(Q_n)\,\mu^{\mathrm{abs}}(\overline{Q}_n)$. Combining these inequalities gives

$$\liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) \geq \frac{\mu^{\mathrm{abs}}(\overline{Q}_n) - \mu^{\mathrm{ess}}(\overline{D})}{\lambda} + \frac{(\overline{Q}_n \cdot \overline{E})}{(Q_n)} - \frac{c\,\lambda}{2\,(Q_n)}.$$

Using (5), we can apply this to a suitable sequence of real numbers $(\lambda_n)_n$ to obtain

$$\liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) \geq \limsup_{n\to\infty} \frac{(\overline{Q}_n \cdot \overline{E})}{(Q_n)},$$

and then (7) by applying this inequality to $-\overline{E}$. This concludes the proof in the one-dimensional case.

In higher dimensions this argument breaks down as a lower bound of the form (8) is not precise enough to take advantage of the condition in Theorem 2. This is where we need the finer estimate from Corollary 5.3.

When specialized to semiabelian varieties, our proof of Theorem 2 is different from Kühne's, although both ultimately rely on an asymptotic use of the arithmetic Siu's inequality: indeed, we do not need to modify the semiabelian variety through a sequence of isogenies as in [Küh22], but rather produce a suitable sequence of semipositive approximations sitting on the given semiabelian variety.

As emphasized in [Che11], the variational principle reduces the equidistribution property to the differentiability of invariants associated to adelic $\mathbb{R}$-divisors. For example, Yuan's theorem is a consequence of the differentiability of the χ-volume function [Yua08] whereas Chen's equidistribution theorem follows from the differentiability of the arithmetic volume function [Che11]. Similarly, our main results concern the differentiability of the essential minimum function, which is equivalent to the height convergence property in (4) and thus implies Theorem 2. We refer the reader to Section 4.1 for a review of the correspondence between differentiability and equidistribution.

Organization. Sections 1 and 2 contain the material we need about $\mathbb{R}$-divisors and adelic $\mathbb{R}$-divisors. In Section 3 we study the relationship between Fujita approximations and positive intersection numbers of adelic $\mathbb{R}$-divisors. We state our main theorem in Section 4 and prove it in Section 5 together with some complements like the partial converse and the logarithmic equidistribution theorem. In Sections 6 and 7 we apply these results in the settings of toric varieties and of dynamical systems, including semiabelian varieties. Finally, in Section 8 we extend our main theorem to quasi-projective varieties. Appendix A contains the auxiliary results in convex analysis that are needed for the application to toric varieties.

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Notation and conventions

We let K be a number field and $\overline{K}$ a fixed algebraic closure of K. We also let $\mathcal{O}_K$ be the ring of integers of K. We denote by $\mathfrak{M}_K$ the set of places of K and by $\mathfrak{M}_K^\infty \subset \mathfrak{M}_K$ the subset of Archimedean places. For each $v \in \mathfrak{M}_K$ we denote by K_v the completion of K with respect to v, and by $\mathbb{Q}_v$ the closure of $\mathbb{Q}$ in K_v. We endow K_v with the unique absolute value $|\cdot|_v$ extending the usual absolute value on $\mathbb{Q}_v$ and set

$$n_v = \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]}.$$

These absolute values and weights verify the *product formula*:

$$\sum_{v \in \mathfrak{M}_K} n_v \log |\alpha|_v = 0 \quad \text{for } \alpha \in K^\times.$$

Furthermore, for any place $v_0 \in \mathfrak{M}_\mathbb{Q}$ we have $\sum_{v|v_0} n_v = 1$, the sum being over the places $v \in \mathfrak{M}_K$ above v_0.

For each $v \in \mathfrak{M}_K$ we fix an algebraic closure $\overline{K}_v$ of K_v, which admits a unique absolute value extending $|\cdot|_v$. We denote by $\mathbb{C}_v$ the completion of $\overline{K}_v$, and we still denote by $|\cdot|_v$ the induced absolute value. We also fix an embedding $\overline{K} \hookrightarrow \overline{K}_v \subset \mathbb{C}_v$.

A *variety* is a separated and integral scheme of finite type over a field. We let X be a normal projective variety over K of dimension $d \geq 1$. A *modification* of X is a birational projective morphism $\phi \colon X' \to X$, that is said *normal* if so is X'. We write $X_{K'} = X \times_K \operatorname{Spec}(K')$ for any field extension $K \subset K'$. The elements of $X(\overline{K})$ are called the *algebraic points* of X.

A *measure* is a Radon measure on a locally compact Hausdorff space. A *signed measure* is a difference of two measures.

1. The inradius of an $\mathbb{R}$-divisor

Throughout the text we assume some working knowledge of $\mathbb{R}$-divisors and their positivity properties as presented in [Laz04, Chapters 1 and 2]. Nevertheless, in this section we recall the basic objects and notations. We pay special attention to the inradius of an $\mathbb{R}$-divisor, explaining its properties and relation to other invariants.

1.1. **Intersection numbers and global sections of $\mathbb{R}$-divisors.** We denote by $\operatorname{Div}(X)$ the Abelian group of Cartier divisors on X and by $\operatorname{Div}(X)_\mathbb{R} = \operatorname{Div}(X) \otimes_\mathbb{Z} \mathbb{R}$ the real vector space of $\mathbb{R}$-Cartier divisors. Since X is normal, there is an injective morphism from $\operatorname{Div}(X)$ to the free Abelian group of Weil divisors of X. Therefore $\operatorname{Div}(X)$ has no torsion and there is an inclusion $\operatorname{Div}(X) \subset \operatorname{Div}(X)_\mathbb{R}$. Since we will be mainly concerned with Cartier divisors and $\mathbb{R}$-Cartier divisors, we just call them divisors and $\mathbb{R}$-divisors for short.

We denote by $\operatorname{Rat}(X)$ the field of rational functions of X and we set $\operatorname{Rat}(X)_\mathbb{R}^\times = \operatorname{Rat}(X)^\times \otimes_\mathbb{Z} \mathbb{R}$. For a nonzero rational function $f \in \operatorname{Rat}(X)^\times$ we denote by $\operatorname{div}(f) \in \operatorname{Div}(X)$ its associated principal divisor. This assignment extends by linearity to a map

$$\operatorname{div} \colon \operatorname{Rat}(X)_\mathbb{R}^\times \longrightarrow \operatorname{Div}(X)_\mathbb{R}.$$

Given two $\mathbb{R}$-divisors D, D' on X we write $D \equiv D'$ when they are linearly equivalent, that is when $D' = D + \operatorname{div}(f)$ for some $f \in \operatorname{Rat}(X)_\mathbb{R}^\times$.

For $D \in \mathrm{Div}(X)_{\mathbb{R}}$ we denote by $[D]$ its associated $\mathbb{R}$-Weil divisor. The *support* of D, denoted by $\mathrm{supp}(D)$, is defined as the support of this $\mathbb{R}$-Weil divisor. It is a closed subset of X.

For an integer $0 \leq r \leq d$, an r-dimensional cycle Z of X and a family of $\mathbb{R}$-divisors $D_i \in \mathrm{Div}(X)_{\mathbb{R}}$, $i = 1, \dots r$, we denote their intersection number by

$$(D_1 \cdots D_r \cdot Z) \in \mathbb{R}.$$

When $Z = X$ we simply write this quantity as $(D_1 \cdots D_d)$. It depends only on the linear equivalence classes of the $\mathbb{R}$-divisors and is invariant under normal modifications. If $D \in \mathrm{Div}(X)_{\mathbb{R}}$ is nef then the intersection number (D^d) coincides with the volume $\mathrm{vol}(D)$ of D.

Given $D_1, \dots, D_{d-1}, D_d, D'_d \in \mathrm{Div}(X)_{\mathbb{R}}$ with D_i nef for $1 \leq i \leq d-1$ and $D_d - D'_d$ pseudo-effective, the corresponding intersection numbers compare as

$$(D_1 \cdots D_{d-1} \cdot D'_d) \leq (D_1 \cdots D_{d-1} \cdot D_d). \tag{1.1}$$

The *space of global sections* of an $\mathbb{R}$-divisor D is defined as

$$\Gamma(X, D) = \{(f, D) \mid f \in \mathrm{Rat}(X)^{\times},\ D + \mathrm{div}(f) \geq 0\} \cup \{0\}.$$

It is a finite dimensional real vector space. Given $s = (f, D) \in \Gamma(X, D) \setminus \{0\}$ we set

$$\mathrm{div}(s) = D + \mathrm{div}(f) \in \mathrm{Div}(X)_{\mathbb{R}}$$

for the corresponding $\mathbb{R}$-divisor in the linear equivalence class of D.

Given global sections $s_1 = (f_1, D_1)$ and $s_2 = (f_2, D_2)$ of $\mathbb{R}$-divisors $D_1, D_2 \in \mathrm{Div}(X)_{\mathbb{R}}$, their product is defined as

$$s_1 \otimes s_2 = (f_1 f_2, D_1 + D_2) \in \Gamma(X, D_1 + D_2).$$

The *algebra of sections* of D is then defined as the direct sum

$$R(D) = \bigoplus_{m \in \mathbb{N}} \Gamma(X, mD)$$

endowed with the structure of graded algebra induced by this product.

1.2. **Definition and basic properties of the inradius.** The notion of inradius of a line bundle with respect to another one was introduced by Tessier as a measure of its bigness [Tei82]. As explained in *loc. cit.*, the terminology is inspired by its interpretation in the toric setting, where it identifies with the usual inradius in the sense of convex geometry (Proposition 6.1).

Definition 1.1. Let P, A be big $\mathbb{R}$-divisors on X. The *inradius* of P with respect to A is defined as

$$r(P; A) = \sup\{\lambda \in \mathbb{R} \mid P - \lambda A \text{ is big}\} = \sup\{\lambda \in \mathbb{R} \mid P - \lambda A \text{ is pseudo-effective}\}.$$

We have $r(P; A) < \infty$: indeed, choosing any ample divisor H then for any $\lambda \in \mathbb{R}$ such that $P - \lambda A$ is pseudo-effective we have $\lambda \leq (H^{d-1} \cdot P)/(H^{d-1} \cdot A)$ by (1.1). On the other hand, since P is big we also have $r(P; A) > 0$ by the continuity of the volume function. Hence $r(P; A)$ is a positive real number.

Note that $P - r(P; A)A$ is pseudo-effective. Moreover

$$r(\delta P; A) = \delta\, r(P; A) \quad \text{and} \quad r(P; \delta A) = \delta^{-1}\, r(P; A) \qquad \text{for } \delta \in \mathbb{R}_{>0},$$

and $r(\phi^* P; \phi^* A) = r(P; A)$ for any normal modification $\phi\colon X' \to X$. This latter property follows from the fact that the volume is invariant under birational morphisms.

Changing the big $\mathbb{R}$-divisor of reference A can only modify the inradius $r(P;A)$ up to a bounded factor, as we next show.

Lemma 1.2. *Let P, A_1, A_2 be big $\mathbb{R}$-divisors on X. Then*

$$r(A_2;A_1)\,r(P;A_2) \le r(P;A_1) \le \frac{1}{r(A_1;A_2)}\,r(P;A_2).$$

Proof. Since both $P - r(P;A_2)A_2$ and $A_2 - r(A_2;A_1)A_1$ are pseudo-effective, so is $P - r(P;A_2)\,r(A_2;A_1)\,A_1$. Therefore $r(A_2;A_1)\,r(P;A_2) \le r(P;A_1)$, which gives the first inequality. The second follows by interchanging the roles of A_1 and A_2. $\square$

The next key lemma shows that the inradius can be controlled up to a constant factor by a quotient of intersections numbers.

Lemma 1.3. *Let P, A be big and nef $\mathbb{R}$-divisors on X. Then*

$$\frac{(P^d)}{d\,(P^{d-1}\cdot A)} \le r(P;A) \le \frac{(P^d)}{(P^{d-1}\cdot A)}.$$

Proof. For $\lambda \in \mathbb{R}$, by Siu's inequality [Laz04, Theorem 2.2.15] we have that $P - \lambda A$ is big whenever

$$(P^d) > \lambda\, d\,(P^{d-1}\cdot A).$$

This gives the first inequality. The second follows from the facts that P is nef and $P - r(P;A)A$ is pseudo-effective, which imply

$$(P^d) - r(P;A)\,(P^{d-1}\cdot A) = (P^{d-1}\cdot (P - r(P;A)A)) \ge 0$$

by the inequality (1.1). $\square$

Lemma 1.4. *Let P, A be big and nef $\mathbb{R}$-divisors on X with $A - P$ pseudo-effective. Then*

$$r(P;A) \ge \frac{(P^d)}{d\,(A^d)}.$$

Proof. Since $A - P$ is pseudo-effective and A, P are nef, we get $(P^{d-1}\cdot A) \le (A^d)$ by iteratively applying (1.1). The result then follows from Lemma 1.3. $\square$

The inradius allows to control the behavior of the volume function with respect to perturbations, under suitable positivity assumptions.

Lemma 1.5. *Let $P, E, A \in \mathrm{Div}(X)_{\mathbb{R}}$ with P, A big and $A \pm E$ pseudo-effective. Then for every $\lambda \in \mathbb{R}$ with $0 \le \lambda \le r(P;A)$ we have*

$$\Big(1 - \frac{\lambda}{r(P;A)}\Big)^d \mathrm{vol}(P) \le \mathrm{vol}(P + \lambda E) \le \Big(1 + \frac{\lambda}{r(P;A)}\Big)^d \mathrm{vol}(P).$$

Proof. Since $A - E$ is pseudo-effective we have that

$$\frac{1}{r(P;A)}P - E = \frac{1}{r(P;A)}(P - r(P;A)A) + A - E$$

is also pseudo-effective. Therefore

$$\mathrm{vol}(P + \lambda E) \le \mathrm{vol}\Big(P + \frac{\lambda}{r(P;A)}P\Big) = \Big(1 + \frac{\lambda}{r(P;A)}\Big)^d \mathrm{vol}(P)$$

because the volume function is positive homogeneous of degree d and increases along pseudo-effective directions. Similarly,

$$\frac{1}{r(P;A)}P+E=\frac{1}{r(P;A)}(P-r(P;A)A)+A+E$$

is pseudo-effective and so

$$\left(1-\frac{\lambda}{r(P;A)}\right)^{d}\operatorname{vol}(P)=\operatorname{vol}\left(P-\frac{\lambda}{r(P;A)}P\right)\leq\operatorname{vol}(P+\lambda E)$$

as stated. □

2. Adelic $\mathbb{R}$-divisors

In this section we recall the definition and basic facts concerning adelic $\mathbb{R}$-divisors, referring to [BPS14, BMPS16, Mor16] for the proofs and more details.

2.1. **First definitions.** For $v\in\mathfrak{M}_K$ we denote by X_v^{an} the analytification of $X_{\mathbb{C}_v}$ in the sense of Berkovich, see [BPS14, Section 1.2] and [Mor16, Section 1.3] for short introductions sufficient for our purposes. There is an injective map $X(\mathbb{C}_v)\hookrightarrow X_v^{\mathrm{an}}$ that induces an inclusion

$$\iota_v\colon X(\overline{K})\longrightarrow X_v^{\mathrm{an}}$$

via the chosen embedding $\overline{K}\hookrightarrow\overline{K}_v\subset\mathbb{C}_v$.

For an algebraic point $x\in X(\overline{K})$ we denote by $O(x)\subset X(\overline{K})$ its orbit under the action of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$, and by

$$O(x)_v=\iota_v(O(x))\subset X_v^{\mathrm{an}}$$

its image under ι_v. It does not depend on the choice of the embedding.

The local Galois group $G_v=\mathrm{Gal}(\overline{K}_v/K_v)$ acts on X_v^{an}. We denote by $C(X_v^{\mathrm{an}})$ the space of continuous real-valued functions on X_v^{an} and by $C(X_v^{\mathrm{an}})^{G_v}\subset C(X_v^{\mathrm{an}})$ the subspace of those functions that are G_v-invariant.

Let $D\in\mathrm{Div}(X)_{\mathbb{R}}$ and $v\in\mathfrak{M}_K$. A *continuous v-adic Green function* for D is a G_v-invariant function

$$g_v\colon X_v^{\mathrm{an}}\setminus\mathrm{supp}(D)_v^{\mathrm{an}}\to\mathbb{R}$$

with the property that for each open subset $U\subset X$ where D is defined by an $\mathbb{R}$-rational function f we have that $g_v+\log|f_v^{\mathrm{an}}|_v$ extends to a continuous function on U_v^{an}, with f_v^{an} the v-adic analytification of f. In this text we only consider continuous v-adic Green functions, and so we call them v-adic Green functions for short.

Let $U\subset\mathrm{Spec}(\mathcal{O}_K)$ be a nonempty open subset. A *model* of X over U is a normal integral projective scheme $\mathcal{X}\to U$ such that $X=\mathcal{X}\times_U\mathrm{Spec}(K)$. For $D\in\mathrm{Div}(X)_{\mathbb{R}}$, a *model* of (X,D) over U is a pair $(\mathcal{X},\mathcal{D})$ where $\mathcal{X}$ is a model of X over U and $\mathcal{D}$ is an $\mathbb{R}$-divisor on $\mathcal{X}$ whose restriction to X coincides with D. Such a model induces a v-adic Green function for D for each place $v\in U$ that we denote by $g_{\mathcal{D},v}$, see [Mor16, Section 2.1] for details.

Definition 2.1. An *adelic $\mathbb{R}$-divisor* on X is a pair $\overline{D}=(D,(g_v)_{v\in\mathfrak{M}_K})$ with $D\in\mathrm{Div}(X)_{\mathbb{R}}$ and g_v a v-adic Green function for D for each $v\in\mathfrak{M}_K$, such that there is a model $(\mathcal{X},\mathcal{D})$ of (X,D) over a nonempty open subset $U\subset\mathrm{Spec}(\mathcal{O}_K)$ with $g_v=g_{\mathcal{D},v}$ for all $v\in U$. When D is a divisor we say that $\overline{D}$ is an *adelic divisor*.

We say that $\overline{D}$ is an adelic $\mathbb{R}$-divisor *over* D and conversely, we say that D is the *geometric $\mathbb{R}$-divisor* of $\overline{D}$.

Unless otherwise stated, given an adelic $\mathbb{R}$-divisor $\overline{D}$ on X we use the same letter D to denote its geometric $\mathbb{R}$-divisor and we denote by $(g_{\overline{D},v})_{v\in\mathfrak{M}_K}$ its family of Green functions.

The set of adelic $\mathbb{R}$-divisors forms a real vector space that we denote by $\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, and the subgroup of adelic divisors is denoted by $\widehat{\mathrm{Div}}(X)$.

Example 2.2. Let $(\varphi_v)_{v\in\mathfrak{M}_K}$ with $\varphi_v \in C(X_v^{\mathrm{an}})^{G_v}$ for each $v \in \mathfrak{M}_K$ and $\varphi_v = 0$ for all except finitely many v. Then $(0, (\varphi_v)_{v\in\mathfrak{M}_K})$ is an adelic divisor over $0 \in \mathrm{Div}(X)$. Every adelic divisor over the zero divisor of X is of this form.

Denote by $[\infty] = (0, (\varphi_v)_{v\in\mathfrak{M}_K}) \in \widehat{\mathrm{Div}}(X)$ the adelic divisor over $0 \in \mathrm{Div}(X)_{\mathbb{R}}$ given by the constant functions $\varphi_v = 1$ if v is Archimedean, and $\varphi_v = 0$ if v is non-Archidemedean. Then for $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and $t \in \mathbb{R}$ we set

$$\overline{D}(t) = \overline{D} - t\,[\infty] \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}. \tag{2.1}$$

Given $f \in \mathrm{Rat}(X)^{\times}$ we set $\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), (-\log|f_v^{\mathrm{an}}|_v)_{v\in\mathfrak{M}_K})$ for its associated principal divisor. This assignment extends by linearity to a map

$$\widehat{\mathrm{div}}\colon \mathrm{Rat}(X)_{\mathbb{R}}^{\times} \longrightarrow \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

Two adelic $\mathbb{R}$-divisors $\overline{D}, \overline{D'} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ are said *linearly equivalent*, denoted $\overline{D} \equiv \overline{D'}$, when there is $f \in \mathrm{Rat}(X)_{\mathbb{R}}^{\times}$ such that $\overline{D'} = \overline{D} + \widehat{\mathrm{div}}(f)$.

For $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and a morphism $\phi\colon X' \to X$ from a normal projective variety X' whose image is not contained in $\mathrm{supp}(D)$, the *pullback* $\phi^*\overline{D} \in \widehat{\mathrm{Div}}(X')_{\mathbb{R}}$ is defined as the $\mathbb{R}$-divisor ϕ^*D equipped at each place $v \in \mathfrak{M}_K$ with the pullback to $(X')_v^{\mathrm{an}}$ of $g_{\overline{D},v}$ by the v-adic analytification of ϕ.

For $D \in \mathrm{Div}(X)_{\mathbb{R}}$ and $v \in \mathfrak{M}_K$ let g_v be a v-adic Green function for D. We say that g_v is *semipositive* when it is of $(C^0 \cap \mathrm{PSH})$-type in the sense of [Mor16, Section 1.4 and Definition 2.1.6]. In the Archimedean case this means that g_v is plurisubharmonic, whereas in the non-Archimedean case it means that g_v can be uniformly approached by the v-adic Green functions of a sequence of vertically nef models. On the other hand, we say that g_v is *DSP* (short for difference of semipositive) if there are semipositive v-adic Green functions $g_{1,v}$ and $g_{2,v}$ such that $g_v = g_{1,v} - g_{2,v}$. Note that if D admits a semipositive v-adic Green function then it is nef, and in particular $\mathrm{vol}(D) = (D^d)$.

Passing to the global situation, an adelic $\mathbb{R}$-divisor $\overline{D}$ is *semipositive* if all its v-adic Green functions are semipositive, and is *DSP* if there are two semipositive adelic $\mathbb{R}$-divisors $\overline{D}_1$, $\overline{D}_2$ such that $\overline{D} = \overline{D}_1 - \overline{D}_2$. We denote by

$$\widehat{\mathrm{DSP}}(X)_{\mathbb{R}} \subset \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$$

the subspace of DSP adelic $\mathbb{R}$-divisors of X.

Remark 2.3. To an adelic divisor $\overline{D} = (D, (g_v)_{v\in\mathfrak{M}_K})$ on X one can associate a metrized line bundle $\overline{L} = (\mathcal{O}_X(D), (\|.\|_v)_{v\in\mathfrak{M}_K})$ on X in the sense of Zhang [Zha95b], and every such metrized line bundle can be constructed in this way [BMPS16, Proposition 3.8]. The metrized line bundle $\overline{L}$ is semipositive in the sense of [Zha95b] if and only if $\overline{D}$ is semipositive, and it is integrable in the sense of [Zha95b] if and only if $\overline{D}$ is DSP.

2.2. **Heights of points and cycles.** Given $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and $x \in X(\overline{K})$, the *height* of x with respect to $\overline{D}$ is defined as

$$h_{\overline{D}}(x) = \sum_{v \in \mathfrak{M}_K} \frac{n_v}{\# O(x)_v} \sum_{y \in O(x)_v} g_{\overline{D'},v}(y) \in \mathbb{R}$$

for any $\overline{D'} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with $\overline{D'} \equiv \overline{D}$ such that $x \notin \mathrm{supp}(D')(\overline{K})$. This quantity does not depend on the choice of $\overline{D'}$ thanks to the product formula.

For an integer $0 \leq r \leq d$ let Z be an r-dimensional cycle of X and $\overline{D}_i$, $i = 1, \dots, r$, a family of adelic $\mathbb{R}$-divisors on X. Let $v \in \mathfrak{M}_K$ and assume that $g_{\overline{D}_i,v}$ is DSP for all i. Then there is a signed measure on X_v^{an}, denoted by

$$c_1(\overline{D}_{1,v}) \wedge \cdots \wedge c_1(\overline{D}_{r,v}) \wedge \delta_{Z_v^{\mathrm{an}}},$$

supported on Z_v^{an} and with total mass $(D_1 \cdots D_r \cdot Z)$. When the D_i's are divisors, it is defined using the complex Monge-Ampère operator in the Archimedean case, whereas in the non-Archimedean case it is the signed measure introduced by Chambert-Loir [Cha06]. This construction extends by multilinearity and continuity to the general case of adelic $\mathbb{R}$-divisors with DSP v-adic Green functions. When these v-adic Green functions are semipositive, it is actually a measure.

In the case when $Z = X$ and $\overline{D}_1 = \cdots = \overline{D}_d = \overline{D}$, this signed measure is called the *v-adic Monge-Ampère measure* of $\overline{D}$ and denoted by $c_1(\overline{D}_v)^{\wedge d}$.

Assume now that $\overline{D}_1, \dots, \overline{D}_r$ are DSP, and let $\overline{D}_{r+1}$ be a further adelic $\mathbb{R}$-divisor on X. The *height* $h_{\overline{D}_1,\dots,\overline{D}_{r+1}}(Z)$ of Z with respect to $\overline{D}_1, \dots, \overline{D}_{r+1}$ is defined by induction on the dimension r of the cycle. When $r = 0$ it is given by linearity from the previous definition of height of points, whereas when $r > 0$ it is given by the *arithmetic Bézout formula*:

$$\begin{aligned} h_{\overline{D}_1,\dots,\overline{D}_{r+1}}(Z) &= h_{\overline{D}_1,\dots,\overline{D}_r}(D' \cdot Z) \\ &\quad + \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{D'},v}\, c_1(\overline{D}_{1,v}) \wedge \cdots \wedge c_1(\overline{D}_{r,v}) \wedge \delta_{Z_v^{\mathrm{an}}} \end{aligned} \tag{2.2}$$

for any $\overline{D'} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with $\overline{D'} \equiv \overline{D}_{r+1}$ such that D' intersects Z properly, where $D' \cdot Z$ is the intersection cycle. This Bézout formula is well-defined because the v-adic Green function $g_{\overline{D'},v}$ is integrable with respect to the signed measure therein [CT09, Theorem 4.1]. Furthermore, it does not depend on the choice of $\overline{D'}$ and it is multilinear in $\overline{D}_1, \dots, \overline{D}_{r+1}$ [BPS14, Section 1.5], [BMPS16, page 225]. If $\overline{D}_{r+1} \in \widehat{\mathrm{DSP}}(X)_{\mathbb{R}}$ this construction is symmetric in $\overline{D}_1, \dots, \overline{D}_{r+1}$. For $\overline{D} \in \widehat{\mathrm{DSP}}(X)_{\mathbb{R}}$ we write $h_{\overline{D}}(Z)$ for the height of Z with respect to $r+1$ copies of $\overline{D}$.

For $\overline{D}_i \in \widehat{\mathrm{DSP}}(X)_{\mathbb{R}}$, $i = 1, \dots, d$, and $\overline{D}_{d+1} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, we define their *arithmetic intersection number* as

$$(\overline{D}_1 \cdots \overline{D}_{d+1}) = h_{\overline{D}_1,\dots,\overline{D}_{d+1}}(X) \in \mathbb{R}.$$

This quantity only depends on the linear equivalence classes of these adelic $\mathbb{R}$-divisors, and for any normal modification $\phi\colon X' \to X$ we have

$$(\phi^*\overline{D}_1 \cdots \phi^*\overline{D}_{d+1}) = (\overline{D}_1 \cdots \overline{D}_{d+1}).$$

It follows from these definitions that for any adelic divisor $\overline{0} = (0, (\varphi_v)_{v\in\mathfrak{M}_K})$ over the zero divisor of X we have

$$(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{0}) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} \varphi_v \; c_1(\overline{D}_{1,v}) \wedge \cdots \wedge c_1(\overline{D}_{d,v}). \tag{2.3}$$

In particular,

$$(\overline{D}_1 \cdots \overline{D}_d \cdot [\infty]) = \sum_{v\in\mathfrak{M}_K^\infty} n_v \int_{X_v^{\mathrm{an}}} c_1(\overline{D}_{1,v}) \wedge \cdots \wedge c_1(\overline{D}_{d,v}) = (D_1 \cdots D_d). \tag{2.4}$$

2.3. **Arithmetic volumes and positivity.** Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Given a nonzero global section $s = (f, D) \in \Gamma(X, D)\backslash\{0\}$, for each point $y \in X_v^{\mathrm{an}}\backslash\mathrm{supp}(\mathrm{div}(s))_v^{\mathrm{an}}$ we set

$$\|s(y)\|_{\overline{D},v} = \exp(-g_{\overline{D}+\widehat{\mathrm{div}}(f),v}(y)).$$

By [Mor16, Propositions 1.4.2 and 2.1.3], this assignment can be uniquely extended to a continuous real-valued function $y \in X_v^{\mathrm{an}} \mapsto \|s(y)\|_{\overline{D},v} \in \mathbb{R}$, called the *$v$-adic norm* of s. We also set

$$\|s\|_{\overline{D},v,\sup} = \sup_{y\in X_v^{\mathrm{an}}} \|s(y)\|_{\overline{D},v}.$$

The height of a point outside the support of a global section can be bounded from below in terms of these sup-norms: for $m \in \mathbb{N}_{>0}$ and $s \in \Gamma(X, mD) \setminus \{0\}$ we have

$$h_{\overline{D}}(x) \geq -\frac{1}{m} \sum_{v\in\mathfrak{M}_K} n_v \log \|s\|_{\overline{D},v,\sup} \quad \text{ for all } x \in X(\overline{K}) \setminus \mathrm{supp}(\mathrm{div}(s))(\overline{K}). \tag{2.5}$$

For $s \in \Gamma(X, D) \setminus \{0\}$ we say that s is *small* if $\|s\|_{\overline{D},v,\sup} \leq 1$ for all $v \in \mathfrak{M}_K$, and by convention we agree that $0 \in \Gamma(X, D)$ is small. We denote by $\widehat{\Gamma}(X, \overline{D})$ the set of small global sections of $\overline{D}$.

Let $\mathbb{A}_K$ be the ring of adèles of K and consider the adelic unit ball

$$\mathbb{B}_{\overline{D}} = \{(s_v)_{v\in\mathfrak{M}_K} \in \Gamma(X, D) \otimes_K \mathbb{A}_K \mid \|s_v\|_{\overline{D},v,\sup} \leq 1 \ \text{ for all } v \in \mathfrak{M}_K\}.$$

We have that $\Gamma(X, D) \otimes_K \mathbb{A}_K$ is a locally compact group and $\Gamma(X, D)$ is a lattice within it *via* the diagonal embedding $K \hookrightarrow \mathbb{A}_K$. We denote by μ the unique Haar measure on $\Gamma(X, D) \otimes_K \mathbb{A}_K$ satisfying

$$\mu((\Gamma(X, D) \otimes_K \mathbb{A}_K)/\Gamma(X, D)) = 1$$

and we set $\widehat{\chi}(X, \overline{D}) = \log(\mu(\mathbb{B}_{\overline{D}}))$.

Definition 2.4. The *arithmetic volume* and the *χ-volume* of $\overline{D}$ are respectively defined as

$$\widehat{\mathrm{vol}}(\overline{D}) = \frac{1}{[K : \mathbb{Q}]} \limsup_{m\to\infty} \frac{\log(\#\widehat{\Gamma}(X, m\overline{D}))}{m^{d+1}/(d + 1)!},$$
$$\widehat{\mathrm{vol}}_\chi(\overline{D}) = \frac{1}{[K : \mathbb{Q}]} \limsup_{m\to\infty} \frac{\widehat{\chi}(X, m\overline{D})}{m^{d+1}/(d + 1)!}.$$

We now recall classical positivity notions for adelic $\mathbb{R}$-divisors.

Definition 2.5. We say that $\overline{D}$ is

(1) *effective* if D is effective and $g_{\overline{D},v} \geq 0$ on $X_v^{\mathrm{an}} \setminus \mathrm{supp}(D)_v^{\mathrm{an}}$ for all $v \in \mathfrak{M}_K$,

(2) *big* if $\widehat{\mathrm{vol}}(\overline{D}) > 0$,

(3) *pseudo-effective* if $\overline{D}+\overline{B}$ is big for every big $\overline{B}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$,
(4) *nef* if $\overline{D}$ is semipositive and $h_{\overline{D}}(x)\geq 0$ for all $x\in X(\overline{K})$.

In the sequel we recall the basic properties of the arithmetic volume and the χ-volume, referring to [Mor16] for the details and proofs.

Remark 2.6. Unlike [Mor16], here we do not assume that X is geometrically irreducible. In this more general situation the Stein factorization shows that there is a finite extension K'/K with the property that the structural morphism $X\to\mathrm{Spec}(K)$ factors through a morphism $X\to\mathrm{Spec}(K')$ and that X is geometrically connected over K'. Since X is normal, it is geometrically irreducible as a variety over K' and all the cited results from *loc. cit.* extend directly to our setting.

For any normal modification $\phi\colon X'\to X$ we have

$$\widehat{\mathrm{vol}}(\phi^*\overline{D})=\widehat{\mathrm{vol}}(\overline{D})\quad\text{and}\quad\widehat{\mathrm{vol}}_\chi(\phi^*\overline{D})=\widehat{\mathrm{vol}}_\chi(\overline{D}),$$

and moreover $\widehat{\mathrm{vol}}(\overline{D})\geq\widehat{\mathrm{vol}}_\chi(\overline{D})$ [Mor16, Section 4.3]. By [Mor16, Theorem 5.2.1] we also have the following continuity property: for every $\overline{E}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$,

$$\widehat{\mathrm{vol}}(\overline{D})=\lim_{\lambda\to 0}\widehat{\mathrm{vol}}(\overline{D}+\lambda\overline{E})\quad\text{and}\quad\widehat{\mathrm{vol}}_\chi(\overline{D})=\lim_{\lambda\to 0}\widehat{\mathrm{vol}}_\chi(\overline{D}+\lambda\overline{E}).$$

When the geometric $\mathbb{R}$-divisor D is big, a sufficiently large shift of $\overline{D}$ is big.

Lemma 2.7. *If D is big then $\overline{D}+c\,[\infty]$ is big for any sufficiently large $c>0$.*

Proof. For $c\in\mathbb{R}$ we have $\widehat{\chi}(X,\overline{D}+c\,[\infty])=\widehat{\chi}(X,\overline{D})+c\,[K:\mathbb{Q}]\dim_K(\Gamma(X,D))$ by the definition of these quantities. Hence

$$\widehat{\mathrm{vol}}(\overline{D}+c\,[\infty])\geq\widehat{\mathrm{vol}}_\chi(\overline{D}+c\,[\infty])=\widehat{\mathrm{vol}}_\chi(\overline{D})+c\,(d+1)\,\mathrm{vol}(D),$$

which readily implies the statement because $\mathrm{vol}(D)>0$. □

Now let $t\in\mathbb{R}$, and for each $m\in\mathbb{N}$ denote by $R^t_m(\overline{D})$ the K-linear subspace of $\Gamma(X,mD)$ generated by $\widehat{\Gamma}(X,m\overline{D}(t))$ with $\overline{D}(t)=\overline{D}-t[\infty]$ as in (2.1). Note that $\widehat{\Gamma}(X,m\overline{D}(t))$ is the set of global sections $s\in\Gamma(X,mD)$ such that

$$\log\|s\|_{\overline{D},v,\sup}\leq\begin{cases}-mt & \text{if } v \text{ is Archimedean,}\\ 0 & \text{otherwise.}\end{cases}$$

Then we set

$$R^t(\overline{D})=\bigoplus_{m\in\mathbb{N}}R^t_m(\overline{D}),$$

that is a graded subalgebra of the algebra of sections $R(D)$. Its volume is the quantity

$$\mathrm{vol}(R^t(\overline{D}))=\limsup_{m\to\infty}\frac{\dim_K R^t_m(\overline{D})}{m^d/d!}.$$

The next theorem is due to Chen, and allows to express the arithmetic volume of a big adelic $\mathbb{R}$-divisor in terms of the volumes of these graded subalgebras.

Theorem 2.8. *If $\overline{D}$ is big then* $\widehat{\mathrm{vol}}(\overline{D})=(d+1)\displaystyle\int_0^\infty\mathrm{vol}(R^t(\overline{D}))\,dt.$

Proof. When $\overline{D}$ is an adelic divisor, this formula is given by [Che10, Theorem 3.8] (see also [Che11, Formula (5.2)]). It is also a consequence of the results of Boucksom and Chen on arithmetic Okounkov bodies [BC11, Theorems 1.11 and 2.8], whose proof can be carried out for adelic $\mathbb{R}$-divisors using the extension of this theory in [Mor16, Section 7.3]. $\square$

Arithmetic (χ-)volumes coincide with arithmetic intersection numbers under suitable positivity conditions.

Theorem 2.9 ([Mor16, Theorem 5.3.2]). *If $\overline{D}$ is semipositive then*

$$\widehat{\mathrm{vol}}_\chi(\overline{D}) = (\overline{D}^{d+1}).$$

If moreover $\overline{D}$ is nef, then $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}_\chi(\overline{D}) = (\overline{D}^{d+1})$.

The existence of arithmetic Fujita approximations for adelic divisors was established independently by Yuan [Yua09] and Chen [Che10]. We will use the following extension to adelic $\mathbb{R}$-divisors from [Mor16, Theorem 5.1.6].

Theorem 2.10. *Assume that $\overline{D}$ is big. For each $\varepsilon > 0$ there exists a normal modification $\phi\colon X' \to X$ and a nef adelic $\mathbb{R}$-divisor $\overline{P}$ on X' such that $\phi^*\overline{D} - \overline{P}$ is pseudo-effective and*

$$(\overline{P}^{d+1}) = \widehat{\mathrm{vol}}(\overline{P}) \geq \widehat{\mathrm{vol}}(\overline{D}) - \varepsilon.$$

Yuan's arithmetic analogue of Siu's inequality is a key ingredient for equidistribution results in Arakelov geometry [Yua08, Theorem 2.2]. We will use the next extension to adelic $\mathbb{R}$-divisors, which follows from the original one by the continuity of the χ-volume function as in [CM15, Proof of Theorem 7.5].

Theorem 2.11. *Let $\overline{P}_1, \overline{P}_2$ be nef adelic $\mathbb{R}$-divisors on X. Then*

$$\widehat{\mathrm{vol}}_\chi(\overline{P}_1 - \overline{P}_2) \geq (\overline{P}_1^{d+1}) - (d+1)\,(\overline{P}_1^{d} \cdot \overline{P}_2).$$

We now turn to other positivity aspects of adelic $\mathbb{R}$-divisors. First note that if there is $s \in \widehat{\Gamma}(X, \overline{D}) \setminus \{0\}$ then for every big $\overline{B} \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$ and $m \in \mathbb{N}_{>0}$ we have an inclusion

$$\widehat{\Gamma}(X, m\overline{B}) \longhookrightarrow \widehat{\Gamma}(X, m(\overline{D} + \overline{B}))$$

given by the multiplication by $s^{\otimes m}$. Hence $\widehat{\mathrm{vol}}(\overline{D}+\overline{B}) \geq \widehat{\mathrm{vol}}(\overline{B}) > 0$ and $\overline{D}$ is pseudo-effective. In particular, an effective adelic $\mathbb{R}$-divisor is pseudo-effective. Moreover, a nef adelic $\mathbb{R}$-divisor is also pseudo-effective [Mor16, Proposition 4.4.2(2)].

We need the following version of the well-known fact that adelic $\mathbb{R}$-divisors can be approximated by DSP adelic $\mathbb{R}$-divisors.

Lemma 2.12. *Let $\overline{E} \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$. Then for each $\varepsilon > 0$ there exists $\overline{E}' \in \widehat{\mathrm{DSP}}(X)_\mathbb{R}$ over E such that $\overline{E} - \overline{E}'$ is effective and $\overline{E}' + \varepsilon[\infty] - \overline{E}$ is pseudo-effective.*

Proof. By [Mor16, Theorem 4.1.3] there is a finite set $S \subset \mathfrak{M}_K$ containing $\mathfrak{M}_K^\infty$ such that for each $\varepsilon' > 0$ there is a model $(\mathcal{X}, \mathcal{E})$ of (X, E) over $\mathrm{Spec}(\mathcal{O}_K)$ such that setting $g_{\overline{E}',v} = g_{\mathcal{E},v}$ and $\varphi_v = g_{\overline{E},v} - g_{\overline{E}',v}$ for every non-Archimedean place v we have $\varphi_v = 0$ for $v \notin S$ and $0 \leq \varphi_v \leq \varepsilon'$ for $v \in S \setminus \mathfrak{M}_K^\infty$. Applying the Stone-Weierstrass theorem one can also show that for every Archimedean place v there is a smooth v-adic Green function $g_{\overline{E}',v}$ on E such that $\varphi_v = g_{\overline{E},v} - g_{\overline{E}',v}$ also satisfies $0 \leq \varphi_v \leq \varepsilon'$.

By construction, $\overline{E}' = (E, (g_{\overline{E}',v})_{v\in\mathfrak{M}_K})$ is a DSP adelic $\mathbb{R}$-divisor over E and $\overline{E}-\overline{E}' = (0, (\varphi_v)_{v\in\mathfrak{M}_K})$ is effective. Moreover, taking ε' sufficiently small with respect to ε we get from [BMPS16, Lemma 1.11] that a sufficiently high multiple of

$$\overline{E}' + \varepsilon[\infty] - \overline{E} = \varepsilon[\infty] - (0, (\varphi_v)_v)$$

has a nonzero small global section. Thus $\overline{E}' + \varepsilon[\infty] - \overline{E}$ is pseudo-effective. □

Lemma 2.13. *Let $\overline{D}_1, \dots, \overline{D}_d$ be nef adelic $\mathbb{R}$-divisors on X. For every $\overline{D}_{d+1}, \overline{D'}_{d+1} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with $\overline{D}_{d+1} - \overline{D'}_{d+1}$ pseudo-effective we have*

$$(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{D'}_{d+1}) \leq (\overline{D}_1 \cdots \overline{D}_d \cdot \overline{D}_{d+1}).$$

Proof. Setting $\overline{E} = \overline{D}_{d+1} - \overline{D'}_{d+1}$, the inequality is equivalent to the fact that

$$(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{E}) \geq 0.$$

The case when $\overline{E}$ is DSP is given by [Mor16, Proposition 4.5.4(3)]. For the general case, let $\varepsilon > 0$ and take $\overline{E}'$ as in Lemma 2.12. Then $\overline{E} - \overline{E}'$ is effective and $\overline{E}' + \varepsilon[\infty]$ is both DSP and pseudo-effective, and so from the formulae (2.3) and (2.4) we get

$$(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{E}) \geq (\overline{D}_1 \cdots \overline{D}_d \cdot \overline{E}') \geq (\overline{D}_1 \cdots \overline{D}_d \cdot (-\varepsilon[\infty])) = -\varepsilon\,(D_1 \cdots D_d).$$

The result follows by letting $\varepsilon \to 0$. □

We will also need the following auxiliary result.

Lemma 2.14. *Let $\overline{E} \in \widehat{\mathrm{DSP}}(X)_{\mathbb{R}}$. Then there exist big and nef $\overline{N}_1, \overline{N}_2 \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ such that $\overline{E} = \overline{N}_1 - \overline{N}_2$.*

Proof. By definition, there are semipositive $\overline{D}_1, \overline{D}_2 \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ such that $\overline{E} = \overline{D}_1 - \overline{D}_2$. Take any semipositive $\overline{A} \in \mathrm{Div}(X)_{\mathbb{R}}$ with A ample. Then both D_1 and D_2 are nef, and so both $A + D_1$ and $A + D_2$ are ample. It follows from the inequality (2.5) that the height functions

$$h_{\overline{D}_1+\overline{A}}, h_{\overline{D}_2+\overline{A}} \colon X(\overline{K}) \longrightarrow \mathbb{R}$$

are bounded from below by a real number. Taking a sufficiently large $t \in \mathbb{R}$ and letting $\overline{N}_i = \overline{D}_i + \overline{A} + t\,[\infty]$, $i = 1, 2$, we have that $\overline{N}_i$ is nef, and it is big by Lemma 2.7. Since $\overline{E} = \overline{N}_1 - \overline{N}_2$, the lemma is proven. □

2.4. **Absolute and essential minima.** Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Its *absolute minimum* is

$$\mu^{\mathrm{abs}}(\overline{D}) = \inf_{x\in X(\overline{K})} h_{\overline{D}}(x).$$

Clearly $\mu^{\mathrm{abs}}(\phi^*\overline{D}) = \mu^{\mathrm{abs}}(\overline{D})$ for any surjective morphism $\phi\colon X' \to X$ with X' projective and normal. We also have $\mu^{\mathrm{abs}}(\overline{D}) > -\infty$ whenever D is semiample.

By definition, $\overline{D}$ is nef if and only if it is semipositive and $\mu^{\mathrm{abs}}(\overline{D}) \geq 0$. Note that for $t \in \mathbb{R}$ we have

$$h_{\overline{D}(t)}(x) = h_{\overline{D}}(x) - t \quad \text{for } x \in X(\overline{K}),$$

and so when $\overline{D}$ is semipositive we have

$$\mu^{\mathrm{abs}}(\overline{D}) = \sup\{t \in \mathbb{R} \mid \overline{D}(t) \text{ is nef}\}. \tag{2.6}$$

The following lower bound for the height of effective cycles is a consequence of Zhang's theorem on minima [Zha95a, Theorem 5.2].

Lemma 2.15. *Let Z be an effective cycle of X of dimension r. If $\overline{D}$ is semipositive then*
$$h_{\overline{D}}(Z) \geq (r+1)\,\mu^{\mathrm{abs}}(\overline{D})\,(D^r \cdot Z).$$
In particular, if $\overline{D}$ is nef then $h_{\overline{D}}(Z) \geq 0$.

Proof. By linearity we may assume that Z is a subvariety. When D is ample, the result follows then from [Bal24, Corollary 2.9]. For the general case, choose $\overline{A} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ semipositive with A ample. Then for any $\varepsilon > 0$ we have that $D + \varepsilon A$ is ample and so
$$\begin{aligned} h_{\overline{D}+\varepsilon\overline{A}}(Z) &\geq (r+1)\,\mu^{\mathrm{abs}}(\overline{D}+\varepsilon\overline{A})\,((D+\varepsilon A)^r \cdot Z) \\ &\geq (r+1)\,(\mu^{\mathrm{abs}}(\overline{D}) + \varepsilon\mu^{\mathrm{abs}}(\overline{A}))\,((D+\varepsilon A)^r \cdot Z). \end{aligned}$$
We conclude by letting $\varepsilon \to 0$ and using multilinearity. □

The *essential minimum* of $\overline{D}$ is the quantity
$$\mu^{\mathrm{ess}}(\overline{D}) = \sup_{Y \subsetneq X} \inf_{x \in (X \setminus Y)(\overline{K})} h_{\overline{D}}(x),$$
where the supremum is over the closed subsets $Y \subsetneq X$. We have $\mu^{\mathrm{ess}}(\phi^*\overline{D}) = \mu^{\mathrm{ess}}(\overline{D})$ for any dominant and generically finite morphism $\phi\colon X' \to X$ with X' projective and normal [BPS15, Proposition 3.4]. We also have $\mu^{\mathrm{ess}}(\overline{D}) < \infty$, and $\mu^{\mathrm{ess}}(\overline{D}) \in \mathbb{R}$ whenever $R(D) \neq \{0\}$ [BC11, Proposition 2.6].

Lemma 2.16. *Let $\overline{D}_1, \overline{D}_2 \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Then*
(1) $\mu^{\mathrm{ess}}(\overline{D}_1 + \overline{D}_2) \geq \mu^{\mathrm{ess}}(\overline{D}_1) + \mu^{\mathrm{ess}}(\overline{D}_2)$,
(2) if D_1 is big then $\lim_{\lambda\to 0} \mu^{\mathrm{ess}}(\overline{D}_1 + \lambda\overline{D}_2) = \mu^{\mathrm{ess}}(\overline{D}_1)$,
(3) for every $t \in \mathbb{R}$ such that $R^t(\overline{D}_1) \neq \{0\}$ we have $\mu^{\mathrm{ess}}(\overline{D}_1) \geq t$,
(4) if D_1 is big and $\overline{D}_1 - \overline{D}_2$ is pseudo-effective then $\mu^{\mathrm{ess}}(\overline{D}_1) \geq \mu^{\mathrm{ess}}(\overline{D}_2)$.

Proof. The first two points can be found in [Bal21, Lemma 3.15], whereas the third is a direct consequence of the inequality (2.5).

For (4) choose $\overline{B} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ big and $\varepsilon > 0$. Then $\overline{D}_1 + \varepsilon\overline{B} - \overline{D}_2$ is big and therefore $R^0(\overline{D}_1 + \varepsilon\overline{B} - \overline{D}_2) \neq 0$. We deduce from (1) and (3)
$$\mu^{\mathrm{ess}}(\overline{D}_1 + \varepsilon\overline{B}) \geq \mu^{\mathrm{ess}}(\overline{D}_1 + \varepsilon\overline{B} - \overline{D}_2) + \mu^{\mathrm{ess}}(\overline{D}_2) \geq \mu^{\mathrm{ess}}(\overline{D}_2)$$
and conclude by letting $\varepsilon \to 0$ and using (2). □

The next result characterizes the essential mimimum of an adelic $\mathbb{R}$-divisor with big geometric $\mathbb{R}$-divisor. It was obtained by the first author [Bal21, Theorem 1.1] as a consequence of a theorem of Ikoma [Iko15], assuming that $\overline{D}$ is semipositive. This condition was later removed thanks to the independent works of Qu and Yin [QY24] and Szachniewicz [Sza23]. The next version follows by combining [Sza23, Lemma 3.3.5 and Theorem 3.3.7].

Theorem 2.17. *We have*
$$\mu^{\mathrm{ess}}(\overline{D}) \leq \sup\{t \in \mathbb{R} \mid \overline{D}(t) \text{ is pseudo-effective}\},$$
with equality if D is big. In that case we also have
$$\mu^{\mathrm{ess}}(\overline{D}) = \sup\{t \in \mathbb{R} \mid \overline{D}(t) \text{ is big}\} = \sup\{t \in \mathbb{R} \mid R^t(\overline{D}) \neq 0\}.$$

We will also use the next variants of Zhang's lower bound for the essential minimum [Zha95a, Theorem 5.2].

Theorem 2.18. *If D is big then*

$$\mu^{\mathrm{ess}}(\overline{D}) \geq \frac{\widehat{\mathrm{vol}}_{\chi}(\overline{D})}{(d+1)\,\mathrm{vol}(D)}.$$

Moreover, if $\overline{D}$ is big then

$$\mu^{\mathrm{ess}}(\overline{D}) \geq \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\,\mathrm{vol}(R^0(\overline{D}))} \geq \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\,\mathrm{vol}(D)}.$$

Proof. The first inequality is [Bal21, Theorem 7.2(1)]. The second follows from Theorem 2.8 and Lemma 2.16(3), which imply

$$\widehat{\mathrm{vol}}(\overline{D}) = (d+1)\int_0^{\mu^{\mathrm{ess}}(\overline{D})} \mathrm{vol}(R^t(\overline{D}))dt \leq (d+1)\,\mathrm{vol}(R^0(\overline{D}))\,\mu^{\mathrm{ess}}(\overline{D}).$$

□

Under mild positivity assumptions, the condition that Zhang's lower bound is an equality is equivalent to the fact that the essential and the absolute minima coincide.

Theorem 2.19 ([Bal24, Theorem 6.6])**.** *Assume that $\overline{D}$ is semipositive and that D is big. Then*

$$\mu^{\mathrm{ess}}(\overline{D}) = \frac{(\overline{D}^{d+1})}{(d+1)\,(D^d)} \quad \textit{if and only if} \quad \mu^{\mathrm{ess}}(\overline{D}) = \mu^{\mathrm{abs}}(\overline{D}).$$

2.5. **Positive linear functionals on G_v-invariant functions.** Let $v \in \mathfrak{M}_K$ and recall that $C(X_v^{\mathrm{an}})^{G_v}$ denotes the space of G_v-invariant continuous real-valued functions on X_v^{an}. By the Riesz representation theorem, any positive linear functional on $C(X_v^{\mathrm{an}})$ corresponds to a measure on X_v^{an}. We need the following variant of this result for positive linear functionals on $C(X_v^{\mathrm{an}})^{G_v}$.

Lemma 2.20. *Let $\Lambda\colon C(X_v^{\mathrm{an}})^{G_v} \to \mathbb{R}$ be a positive linear functional.*

(1) There is a unique G_v-invariant measure ν on X_v^{an} such that

$$\Lambda(\varphi) = \int_{X_v^{\mathrm{an}}} \varphi\, d\nu \quad \textit{for all } \varphi \in C(X_v^{\mathrm{an}})^{G_v}. \tag{2.7}$$

(2) If $(\nu_n)_n$ is a sequence of G_v-invariant measures on X_v^{an} such that

$$\lim_{n\to\infty} \int_{X_v^{\mathrm{an}}} \varphi\, d\nu_n = \Lambda(\varphi) \quad \textit{for all } \varphi \in C(X_v^{\mathrm{an}})^{G_v},$$

then $\lim_{n\to\infty} \nu_n = \nu$.

Proof. For (1), let $V \subset C(X_v^{\mathrm{an}})$ be the subspace of continuous real-valued functions on X_v^{an} that are $\mathrm{Gal}(\overline{K}_v/K'_v)$-invariant for some finite extension K'/K. By [Yua08, "Equivalence" at page 638] (see also [GM22, Proposition 2.11 and Theorem 2.13]) this subspace is dense in $C(X_v^{\mathrm{an}})$ with respect to the supremum norm.

For each $\varphi \in V$ let K'/K be a finite extension such that φ is $\mathrm{Gal}(\overline{K}_v/K'_v)$-invariant and set

$$\widetilde{\varphi} = \frac{1}{\#\,\mathrm{Gal}(K'_v/K_v)} \sum_{\sigma\in\mathrm{Gal}(K'_v/K_v)} \sigma^*\varphi.$$

This is an element of $C(X_v^{\mathrm{an}})^{G_v}$ that does not depend on the choice of this finite extension. We extend Λ to a G_v-invariant positive linear functional $\widetilde{\Lambda}\colon V \to \mathbb{R}$ by setting

$\widetilde{\Lambda}(\varphi) = \Lambda(\widetilde{\varphi})$, and then by continuity to a G_v-invariant positive linear functional on $C(X_v^{\mathrm{an}})$. By the Riesz representation theorem there is a G_v-invariant measure ν on X_v^{an} that satisfies (2.7).

If ν' is another G_v-invariant measure on X_v^{an} satisfying this equality on $C(X_v^{\mathrm{an}})^{G_v}$ then it also does on V. Therefore $\nu' = \nu$ by the density of this subspace.

To prove (2), for each $\varphi \in V$ we have

$$\int_{X_v^{\mathrm{an}}} \varphi \, d\nu_n = \int_{X_v^{\mathrm{an}}} \widetilde{\varphi} \, d\nu_n \xrightarrow[n\to\infty]{} \int_{X_v^{\mathrm{an}}} \widetilde{\varphi} \, d\nu = \int_{X_v^{\mathrm{an}}} \varphi \, d\nu$$

by the G_v-invariance of the measures ν_n, $n \in \mathbb{N}$, and ν. The statement then follows from the density of V. □

Finally we recall a basic instance of Lemma 2.20(1). For $\varphi \in C(X_v^{\mathrm{an}})^{G_v}$ we have that φ is a v-adic Green function over the zero divisor of X, and we define the adelic divisor $\overline{0}^{\varphi}$ by equipping this divisor with φ at v and the zero function at every other place. The assignment

$$\varphi \longmapsto \overline{0}^{\varphi} \tag{2.8}$$

gives an inclusion $C(X_v^{\mathrm{an}})^{G_v} \hookrightarrow \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Now if $\overline{D}$ is a semipositive adelic $\mathbb{R}$-divisor on X, for each $\varphi \in C(X_v^{\mathrm{an}})^{G_v}$ we have

$$(\overline{D}^{d} \cdot \overline{0}^{\varphi}) = n_v \int_{X_v^{\mathrm{an}}} \varphi \, c_1(\overline{D}_v)^{\wedge d} \tag{2.9}$$

by the formula in (2.3). Hence the positive linear functional $C(X_v^{\mathrm{an}})^{G_v} \to \mathbb{R}$ defined by $\varphi \mapsto (\overline{D}^{d} \cdot \overline{0}^{\varphi})$ is represented by the G_v-invariant measure $n_v \, c_1(\overline{D}_v)^{\wedge d}$.

3. Fujita approximations and positive intersection numbers

In this section we revisit Chen's theorem on the differentiability of the arithmetic volume [Che11] to express the derivative of this function at a big adelic $\mathbb{R}$-divisor in terms of arithmetic Fujita approximations. This will allow us to characterize in terms of such approximations the arithmetic positive intersection numbers that are relevant to our results.

3.1. **Differentiability of concave functions.** We first recall the notion of differentiability of functions on real vector spaces and its relation to concavity. Our ambient will be a real vector space V endowed with the topology defined by declaring that a subset $U \subset V$ is open whenever its restriction to any finite-dimensional affine subspace $H \subset V$ is open with respect to the Euclidean topology of H.

Definition 3.1. Let $\Phi\colon U \to \mathbb{R}$ be a function on an open subset $U \subset V$. For a point $x \in U$ and a linear subspace $W \subset V$ we say that Φ is *differentiable at x along W* if for all $z \in W$ the one-sided derivative

$$\partial_z \Phi(x) = \lim_{\lambda \to 0^+} \frac{\Phi(x + \lambda\, z) - \Phi(x)}{\lambda}$$

exists in $\mathbb{R}$ and the map $z \in W \to \partial_z \Phi(x) \in \mathbb{R}$ is linear. When $W = V$, we simply say that Φ is differentiable at x.

A real-valued function Φ on a subset $U \subset V$ is *concave* when U is convex and

$$\Phi(\lambda x + (1-\lambda) y) \geq \lambda \Phi(x) + (1-\lambda)\Phi(y) \quad \text{for all } x, y \in U \text{ and } 0 \leq \lambda \leq 1.$$

Lemma 3.2. *Let $\Phi\colon U \to \mathbb{R}$ be a concave function on an open convex subset of V and let $x \in U$. Then*

(1) for all $z \in V$ the one-sided derivative $\partial_z \Phi(x)$ exists in $\mathbb{R}$ and $\partial_z \Phi(x) \leq -\partial_{-z}\Phi(x)$,

(2) the function Φ is differentiable at x along a linear subspace $W \subset V$ if and only if $\partial_z \Phi(x) = -\partial_{-z}\Phi(x)$ for all $z \in W$.

Proof. For (1) let $a > 0$ be a real number sufficiently small so that $x + \lambda z \in U$ for all $-a < \lambda < a$. Let $\iota\colon (-a, a) \to V$ be the inclusion map defined as $\iota(\lambda) = x + \lambda z$, so that the function $\iota^*\Phi\colon (-a, a) \to \mathbb{R}$ is concave. Then the functions $r_-, r_+\colon (0, a) \to \mathbb{R}$ respectively defined as

$$r_-(\lambda) = \frac{\iota^*\Phi(-\lambda) - \iota^*\Phi(0)}{-\lambda} \quad \text{and} \quad r_+(\lambda) = \frac{\iota^*\Phi(\lambda) - \iota^*\Phi(0)}{\lambda}$$

verify that r_- is non-decreasing, r_+ is non-increasing, and $r_-(\lambda) \geq r_+(\lambda)$ for every λ. Hence both converge when $\lambda \to 0^+$ and

$$-\partial_{-z}\Phi(x) = \lim_{\lambda \to 0^+} r_-(\lambda) \geq \lim_{\lambda \to 0^+} r_+(\lambda) = \partial_z \Phi(x).$$

For (2) it is clear that the differentiability of Φ at x along W implies that $\partial_z \Phi(x) = -\partial_{-z}\Phi(x)$ for all $z \in W$. Conversely, let $z_1, z_2 \in W$ and suppose that this condition holds for these two vectors. For $\lambda > 0$ small we deduce from the concavity of Φ that

$$\frac{\Phi(x + \lambda(z_1 + z_2)) - \Phi(x)}{\lambda} \geq \frac{\Phi(x + 2\lambda z_1) - \Phi(x)}{2\lambda} + \frac{\Phi(x + 2\lambda z_2) - \Phi(x)}{2\lambda}.$$

Letting $\lambda \to 0^+$ we get $\partial_{z_1+z_2}\Phi(x) \geq \partial_{z_1}\Phi(x) + \partial_{z_2}\Phi(x)$. Applying this inequality to $-z_1, -z_2$ together with (1) we obtain

$$-\partial_{-z_1}\Phi(x) - \partial_{-z_2}\Phi(x) \geq -\partial_{-z_1-z_2}\Phi(x) \geq \partial_{z_1+z_2}\Phi(x) \geq \partial_{z_1}\Phi(x) + \partial_{z_2}\Phi(x).$$

By assumption the extremes in these inequalities coincide, and so $\partial_{z_1+z_2}\Phi(x) = \partial_{z_1}\Phi(x) + \partial_{z_2}\Phi(x)$. In addition, the one-sided derivative is positive homogeneous of degree 1 and so linear, as stated. □

3.2. **The differentiability of the arithmetic volume.** Let $\overline{D}$ be a big adelic $\mathbb{R}$-divisor on X.

Definition 3.3. A *Fujita approximation sequence* of $\overline{D}$ is a sequence $(\phi_n, \overline{P}_n)_n$ satisfying the following conditions:

(1) $\phi_n\colon X_n \to X$ is a normal modification,

(2) $\overline{P}_n$ is a nef adelic $\mathbb{R}$-divisor on X_n with $\phi^*\overline{D} - \overline{P}_n$ pseudo-effective,

(3) $\lim_{n\to\infty} (\overline{P}_n^{d+1}) = \widehat{\mathrm{vol}}(\overline{D})$.

The existence of Fujita approximation sequences is warranted by Theorem 2.10. The next result is a variant of Chen's differentiability theorem [Che11] and shows that the derivative of the arithmetic volume function can be realized in terms of any such sequence. Its proof follows closely that of Chen.

Theorem 3.4. *The arithmetic volume function is differentiable at $\overline{D}$, and for any Fujita approximation sequence $(\phi_n, \overline{P}_n)_n$ of $\overline{D}$ we have*

$$\partial_{\overline{E}} \widehat{\mathrm{vol}}(\overline{D}) = (d+1) \lim_{n\to\infty} (\overline{P}_n^d \cdot \phi_n^* \overline{E}) \quad \textit{for all } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

We need the following consequence of the arithmetic Siu's inequality (Theorem 2.11).

Lemma 3.5. *Let $\overline{P}$ and $\overline{E}$ be two adelic $\mathbb{R}$-divisors on X with $\overline{P}$ nef, and $\overline{A}$ a big and nef adelic $\mathbb{R}$-divisor on X such that $\overline{A}-\overline{P}$ is pseudo-effective and $\overline{A}\pm\overline{E}$ are nef. There exists a constant c_d depending only on d such that*

$$\widehat{\mathrm{vol}}(\overline{P}+\lambda\overline{E}) \geq (\overline{P}^{d+1}) + (d+1)\,(\overline{P}^{d}\cdot\overline{E})\,\lambda - c_d\,\widehat{\mathrm{vol}}(\overline{A})\,\lambda^2 \quad \textit{for all } \lambda\in[0,1].$$

Proof. This is given by [Iko15, Proposition 5.1] in the case when all non-Archimedean Green functions are induced by an integral model. The general case follows from this by continuity. □

Proof of Theorem 3.4. Consider the function $\Phi=\widehat{\mathrm{vol}}^{\frac{1}{d+1}}$ on the big cone of $\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. It is positive homogeneous of degree 1 and super-additive by the Brunn-Minkowski inequality, and its domain is an open cone [Mor16, Theorems 5.2.1 and 5.3.1]. Therefore it is a concave function on a convex open subset of a real vector space.

Let $\overline{E}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Applying Lemma 3.2 to this concave function and noting that $\widehat{\mathrm{vol}}=\Phi^{d+1}$ we deduce that the one-sided derivative $\partial_{\overline{E}}\,\widehat{\mathrm{vol}}(\overline{D})$ exists in $\mathbb{R}$ and moreover $-\partial_{-\overline{E}}\,\widehat{\mathrm{vol}}(\overline{D})\geq\partial_{\overline{E}}\,\widehat{\mathrm{vol}}(\overline{D})$. We claim that

$$\partial_{\overline{E}}\,\widehat{\mathrm{vol}}(\overline{D}) \geq (d+1)\,\limsup_{n\to\infty}\,(\overline{P}_n^{d}\cdot\phi_n^*\overline{E}). \tag{3.1}$$

To prove this, we first assume that $\overline{E}$ is DSP. By Lemma 2.14 there is a big and nef adelic $\mathbb{R}$-divisor $\overline{A}$ on X such that $\overline{A}\pm\overline{E}$ are nef. Replacing $\overline{A}$ by a sufficiently large multiple we can also assume that $\overline{A}-\overline{D}$ is pseudo-effective, which implies that $\phi_n^*\overline{A}-\overline{P}_n$ is pseudo-effective for all n. By Lemma 3.5 there exists $c>0$ such that

$$\widehat{\mathrm{vol}}(\overline{D}+\lambda\overline{E}) \geq \widehat{\mathrm{vol}}(\overline{P}_n+\lambda\,\phi_n^*\overline{E}) \geq (\overline{P}_n^{d+1}) + (d+1)\lambda\,(\overline{P}_n^{d}\cdot\phi_n^*\overline{E}) - c\,\lambda^2$$

for all $0<\lambda\leq 1$ and $n\in\mathbb{N}$. Taking the lim sup with respect to n we deduce

$$\frac{\widehat{\mathrm{vol}}(\overline{D}+\lambda\overline{E})-\widehat{\mathrm{vol}}(\overline{D})}{\lambda} \geq (d+1)\,\limsup_{n\to\infty}\,(\overline{P}_n^{d}\cdot\phi_n^*\overline{E}) - c\,\lambda,$$

and we obtain (3.1) in this case by letting $\lambda\to 0$.

For the general case, let $\varepsilon>0$. By Lemma 2.12 there is $\overline{E}'\in\widehat{\mathrm{DSP}}(X)_{\mathbb{R}}$ such that $\overline{E}-\overline{E}'$ and $\overline{E}'+\varepsilon[\infty]-\overline{E}$ are pseudo-effective. Then $\partial_{\overline{E}}\,\widehat{\mathrm{vol}}(\overline{D})\geq\partial_{\overline{E}'}\,\widehat{\mathrm{vol}}(\overline{D})$, and using Lemma 2.13, the formula in (2.4) and the fact that $(P_n^d)=\mathrm{vol}(P_n)$ we obtain

$$(\overline{P}_n^{d}\cdot\phi_n^*\overline{E}') \geq (\overline{P}_n^{d}\cdot\phi_n^*(\overline{E}-\varepsilon[\infty])) = (\overline{P}_n^{d}\cdot\phi_n^*\overline{E}) - \varepsilon\,\mathrm{vol}(P_n) \geq (\overline{P}_n^{d}\cdot\phi_n^*\overline{E}) - \varepsilon\,\mathrm{vol}(D)$$

for all n. From the DSP case we deduce

$$\partial_{\overline{E}}\,\widehat{\mathrm{vol}}(\overline{D}) \geq (d+1)\,\Big(\limsup_{n\to\infty}\,(\overline{P}_n^{d}\cdot\phi_n^*\overline{E}) - \varepsilon\,\mathrm{vol}(D)\Big),$$

and so (3.1) follows by letting $\varepsilon\to 0$. Applying this inequality to $-\overline{E}$ we obtain

$$(d+1)\,\liminf_{n\to\infty}\,(\overline{P}_n^{d}\cdot\phi_n^*\overline{E}) \geq -\partial_{-\overline{E}}\,\widehat{\mathrm{vol}}(\overline{D}) \geq \partial_{\overline{E}}\,\widehat{\mathrm{vol}}(\overline{D}) \geq (d+1)\,\limsup_{n\to\infty}\,(\overline{P}_n^{d}\cdot\phi_n^*\overline{E}).$$

This implies that $\partial_{\overline{E}}\,\widehat{\mathrm{vol}}(\overline{D})=(d+1)\,\lim_{n\to\infty}(\overline{P}_n^{d}\cdot\phi_n^*\overline{E})$, including the fact that this limit exists in $\mathbb{R}$ and that $\widehat{\mathrm{vol}}$ is differentiable at $\overline{D}$. □

Combining this with Chen's formula for the arithmetic volume (Theorem 2.8) we obtain useful information about the asymptotics of Fujita approximation sequences.

Proposition 3.6. *Let $(\phi_n, \overline{P}_n)_n$ be a Fujita approximation sequence of $\overline{D}$. Then*

$$\lim_{n\to\infty}(\overline{P}_n^d \cdot \overline{D}) = \widehat{\mathrm{vol}}(\overline{D}), \quad \lim_{n\to\infty}(P_n^d) = \mathrm{vol}(R^0(\overline{D})), \quad \lim_{n\to\infty}\mu^{\mathrm{ess}}(\overline{P}_n) = \mu^{\mathrm{ess}}(\overline{D}).$$

Proof. The first formula follows from Theorem 3.4 after noting that $\partial_{\overline{D}}\widehat{\mathrm{vol}}(\overline{D}) = (d+1)\widehat{\mathrm{vol}}(\overline{D})$ by the positive homogeneity of the arithmetic volume.

For the second, for all $\lambda > -\mu^{\mathrm{ess}}(\overline{D})$ we have that $\overline{D} + \lambda[\infty]$ is big and

$$\widehat{\mathrm{vol}}(\overline{D} + \lambda[\infty]) = (d+1)\int_0^\infty \mathrm{vol}(R^t(\overline{D} + \lambda[\infty]))\,dt = (d+1)\int_{-\lambda}^\infty \mathrm{vol}(R^t(\overline{D}))\,dt$$

by Theorems 2.17 and 2.8. Since the function $t \mapsto \mathrm{vol}(R^t(\overline{D}))$ is non-increasing, this integral formula implies

$$\partial_{[\infty]}\widehat{\mathrm{vol}}(\overline{D}) = (d+1)\lim_{\lambda\to 0^-}\mathrm{vol}(R^\lambda(\overline{D})), \quad -\partial_{-[\infty]}\widehat{\mathrm{vol}}(\overline{D}) = (d+1)\lim_{\lambda\to 0^+}\mathrm{vol}(R^\lambda(\overline{D})).$$

By Theorem 3.4 these two quantities coincide and so $\partial_{[\infty]}\widehat{\mathrm{vol}}(\overline{D}) = (d+1)\,\mathrm{vol}(R^0(\overline{D}))$, and moreover this derivative equals $(d+1)\lim_{n\to\infty}(\overline{P}_n^d \cdot [\infty]) = (d+1)\lim_{n\to\infty}(P_n^d)$.

For the third, assume by contradiction that there exists $\gamma < \mu^{\mathrm{ess}}(\overline{D})$ such that $\mu^{\mathrm{ess}}(\overline{P}_n) \le \gamma$ for an arbitrarily large n and set

$$c = (d+1)\int_\gamma^{\mu^{\mathrm{ess}}(\overline{D})} \mathrm{vol}(R^t(\overline{D}))\,dt.$$

We have $c = \widehat{\mathrm{vol}}(\overline{D}(\gamma)) > 0$ by Theorem 2.17. Since $\overline{D} - \phi_n^*\overline{P}_n$ is pseudo-effective, by Theorem 2.8 and Lemma 2.16(3) we have

$$\widehat{\mathrm{vol}}(\overline{P}_n) = (d+1)\int_0^{\mu^{\mathrm{ess}}(\overline{P}_n)} \mathrm{vol}(R^t(\overline{P}_n))\,dt \le (d+1)\int_0^\gamma \mathrm{vol}(R^t(\overline{D}))\,dt \le \widehat{\mathrm{vol}}(\overline{D}) - c,$$

which contradicts the last condition in Definition 3.3. ☐

Definition 3.7. For each $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, the *arithmetic positive intersection number* $(\langle \overline{D}^d \rangle \cdot \overline{E})$ is defined as

$$(\langle \overline{D}^d \rangle \cdot \overline{E}) = \lim_{n\to\infty}(\overline{P}_n^d \cdot \phi_n^*\overline{E})$$

for any Fujita approximation sequence $(\phi_n, \overline{P}_n)_n$ of $\overline{D}$. By Theorem 3.4 this limit exists and does not depend on the choice of the sequence.

Remark 3.8. Our definition of the quantity $(\langle \overline{D}^d \rangle \cdot \overline{E})$ agrees with that of Chen [Che11] since both coincide with $(d+1)^{-1}\partial_{\overline{E}}\widehat{\mathrm{vol}}(\overline{D})$. The alternative approach from Definition 3.7 is better suited for our purposes.

Remark 3.9. One can similarly adapt the proof of Boucksom, Favre and Jonsson's differentiability theorem [BFJ09] to show that for $D, E \in \mathrm{Div}(X)_{\mathbb{R}}$ with D big, the geometric positive intersection number $(\langle D^{d-1} \rangle \cdot E)$ introduced in *loc. cit.* can be expressed as

$$(\langle D^{d-1} \rangle \cdot E) = \lim_{n\to\infty}(P_n^{d-1} \cdot \phi_n^* E)$$

for any sequence $(\phi_n, P_n)_n$ such that P_n is a nef $\mathbb{R}$-divisor on a normal modification $\phi_n \colon X_n \to X$ with $\phi_n^* D - P_n$ pseudo-effective and $\lim_{n\to\infty}(P_n^d) = \mathrm{vol}(D)$.

With this definition, the first and second formulae in Proposition 3.6 become

$$(\langle \overline{D}^d \rangle \cdot \overline{D}) = \widehat{\mathrm{vol}}(\overline{D}) \quad \text{and} \quad (\langle \overline{D}^d \rangle \cdot [\infty]) = \mathrm{vol}(R^0(\overline{D})). \tag{3.2}$$

The first of them is [Che11, Corollary 4.4] and is usually interpreted as an asymptotic orthogonality for the Fujita approximations of $\overline{D}$.

These arithmetic positive intersections numbers allow to define the linear functional

$$\Omega_{\overline{D}} \colon \widehat{\mathrm{Div}}(X)_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad \overline{E} \longmapsto \frac{(\langle \overline{D}^d \rangle \cdot \overline{E})}{\mathrm{vol}(R^0(\overline{D}))}. \tag{3.3}$$

By Proposition 3.6, for any Fujita approximation sequence $(\phi_n, \overline{P}_n)_n$ of $\overline{D}$ we have

$$\Omega_{\overline{D}}(\overline{E}) = \lim_{n\to\infty} \frac{(\overline{P}_n^d \cdot \phi_n^* \overline{E})}{(P_n^d)} \quad \text{for all } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}. \tag{3.4}$$

Remark 3.10. This linear functional verifies that $\Omega_{\overline{D}}(\overline{E}) \geq 0$ for every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ pseudo-effective, $\Omega_{\overline{D}}(\widehat{\mathrm{div}}(f)) = 0$ for every $f \in \mathrm{Rat}(X)_{\mathbb{R}}^{\times}$ and $\Omega_{\overline{D}}([\infty]) = 1$, as it follows respectively from Lemma 2.13, the invariance of arithmetic intersection numbers with respect to linear equivalence, and the second formula in (3.2). In particular, $\Omega_{\overline{D}}$ is a normalized GVF functional in the sense of [Sza23].

For each $v \in \mathfrak{M}_K$ we have that $\Omega_{\overline{D}}$ induces a positive linear functional on $C(X_v^{\mathrm{an}})^{G_v}$ through the inclusion in (2.8). Hence by Lemma 2.20(1) there is a unique G_v-invariant measure $\omega_{\overline{D},v}$ on X_v^{an} such that

$$\Omega_{\overline{D}}(\overline{0}^{\varphi}) = \frac{(\langle \overline{D}^d \rangle \cdot \overline{0}^{\varphi})}{\mathrm{vol}(R^0(\overline{D}))} = n_v \int_{X_v^{\mathrm{an}}} \varphi \, d\omega_{\overline{D},v} \quad \text{for all } \varphi \in C(X_v^{\mathrm{an}})^{G_v}. \tag{3.5}$$

It follows from the limit formula in (3.4) together with (2.9) that for any Fujita approximation sequence $(\phi_n, \overline{P}_n)_n$ of $\overline{D}$ we have

$$\int_{X_v^{\mathrm{an}}} \varphi \, d\omega_{\overline{D},v} = \lim_{n\to\infty} \int_{X_v^{\mathrm{an}}} \varphi \, d\omega_{n,v} \quad \text{for all } \varphi \in C(X_v^{\mathrm{an}})^{G_v}$$

with $\omega_{n,v}$ the pushforward to X_v^{an} of $c_1(\overline{P}_{n,v})^{\wedge d}/(P_n^d)$. Hence by Lemma 2.20(2) we have $\omega_{\overline{D},v} = \lim_{n\to\infty} \omega_{n,v}$. In particular, $\omega_{\overline{D},v}$ is a probability measure.

4. The differentiability of the essential minimum

The differentiability of the essential minimum at an adelic $\mathbb{R}$-divisor $\overline{D}$ is closely related to the asymptotic behavior of the $\overline{D}$-small generic sequences of algebraic points. After explaining this relation we state our central result, giving the differentiability of this function assuming the existence of suitable semipositive approximations of $\overline{D}$. We then explain how it implies both Yuan's and Chen's equidistribution theorems [Yua08, Che11]. Finally we give a reformulation of this result in terms of arithmetic positive intersection numbers.

Throughout this section we let $\overline{D}$ be an adelic $\mathbb{R}$-divisor on X over a big $\mathbb{R}$-divisor D.

4.1. **The variational approach to limit heights and equidistribution.** As noted in Section 2.4, the essential minimum function

$$\mu^{\mathrm{ess}} \colon \widehat{\mathrm{Div}}(X)_{\mathbb{R}} \longrightarrow \mathbb{R} \cup \{-\infty\}$$

takes finite values on the open cone $C \subset \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ of adelic $\mathbb{R}$-divisors with big geometric $\mathbb{R}$-divisor. Moreover, it is positive homogeneous of degree 1 and super-additive by Lemma 2.16. Therefore it is a concave function on C. By Lemma 3.2, for every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ the one-sided derivative $\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$ exists in $\mathbb{R}$ and we have

$$-\partial_{-\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}).$$

Definition 4.1. A sequence $(x_\ell)_\ell$ in $X(\overline{K})$ is *generic* if for every closed subset $Y \subsetneq X$ there is $\ell_0 \in \mathbb{N}$ such that $x_\ell \notin Y(\overline{K})$ for all $\ell \geq \ell_0$. When this is the case, we say that $(x_\ell)_\ell$ is *$\overline{D}$-small* if

$$\lim_{\ell \to \infty} h_{\overline{D}}(x_\ell) = \mu^{\mathrm{ess}}(\overline{D}).$$

In the sequel, every considered generic sequence of algebraic points lies in $X(\overline{K})$ unless otherwise stated.

By [BPRS19, Proposition 3.2] for every generic sequence $(x_\ell)_\ell$ we have

$$\liminf_{\ell \to \infty} h_{\overline{D}}(x_\ell) \geq \mu^{\mathrm{ess}}(\overline{D}),$$

and moreover there are generic sequences that are $\overline{D}$-small. A similar conclusion holds for the heights of $\overline{D}$-small generic sequences with respect to another adelic $\mathbb{R}$-divisor $\overline{E}$, replacing the quantity $\mu^{\mathrm{ess}}(\overline{D})$ by the one-sided derivative $\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$.

Lemma 4.2. *Let $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. For every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ we have*

$$-\partial_{-\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \limsup_{\ell \to \infty} h_{\overline{E}}(x_\ell) \geq \liminf_{\ell \to \infty} h_{\overline{E}}(x_\ell) \geq \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}).$$

Moreover, there exists a $\overline{D}$-small generic sequence $(x_\ell)_\ell$ such that

$$\lim_{\ell \to \infty} h_{\overline{E}}(x_\ell) = \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}).$$

Proof. For a $\overline{D}$-small generic sequence $(x_\ell)_\ell$ and $\lambda > 0$ we have $\liminf_{\ell \to \infty} h_{\overline{D}+\lambda\overline{E}}(x_\ell) \geq \mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E})$ and so

$$\liminf_{\ell \to \infty} h_{\overline{E}}(x_\ell) = \liminf_{\ell \to \infty} \frac{h_{\overline{D}+\lambda\overline{E}}(x_\ell) - h_{\overline{D}}(x_\ell)}{\lambda} \geq \frac{\mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E}) - \mu^{\mathrm{ess}}(\overline{D})}{\lambda}.$$

Therefore $\liminf_{\ell \to \infty} h_{\overline{E}}(x_\ell) \geq \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$, and we complete the proof of the first statement by applying this to $-\overline{E}$.

We now pass to the second. By homogeneity, after multiplying $\overline{E}$ by a sufficiently small positive constant we assume without loss of generality that $D + E$ is big. For each $n \in \mathbb{N}_{>0}$ we choose a generic sequence $(x_{n,\ell})_\ell$ such that

$$\lim_{\ell \to \infty} h_{\overline{D}+\frac{1}{n}\overline{E}}(x_{n,\ell}) = \mu^{\mathrm{ess}}\Big(\overline{D} + \frac{1}{n}\overline{E}\Big). \tag{4.1}$$

Let H_i, $i \in \mathbb{N}$, be the collection of all the hypersurfaces of X in an arbitrary order. Then for each n we use (4.1) and the genericity of the sequence $(x_{n,\ell})_\ell$ to choose $\ell(n) \in \mathbb{N}$ such that the algebraic point $y_n = x_{n,\ell(n)} \in X(\overline{K})$ satisfies the conditions:

(1) $y_n \notin \bigcup_{i=0}^{n} H_i$,

(2) $h_{\overline{D}+\frac{1}{n}\overline{E}}(y_n) \le \mu^{\mathrm{ess}}(\overline{D}+\frac{1}{n}\overline{E}) + 1/n^2$,

(3) $h_{\overline{D}}(y_n) \ge \mu^{\mathrm{ess}}(\overline{D}) - 1/n^2$.

By (1) the sequence $(y_n)_n$ is generic. Since $D+E$ is big, this implies that there is $c \in \mathbb{R}$ such that $h_{\overline{D}+\overline{E}}(y_n) \ge c$ for all n, thanks to the lower bound (2.5). Then it follows from the condition (2) that

$$h_{\overline{D}}(y_n) = \frac{n}{n-1}\Big(h_{\overline{D}+\frac{1}{n}\overline{E}}(y_n) - \frac{1}{n}h_{\overline{D}+\overline{E}}(y_n)\Big) \le \frac{n}{n-1}\mu^{\mathrm{ess}}\Big(\overline{D}+\frac{1}{n}\overline{E}\Big) + \frac{1}{n(n-1)} - \frac{c}{n-1}.$$

Therefore the sequence $(y_n)_n$ is also $\overline{D}$-small by the continuity of the essential minimum (Lemma 2.16(2)). On the other hand, the conditions (2) and (3) give

$$\begin{aligned}-\frac{1}{n^2} + \frac{1}{n}h_{\overline{E}}(y_n) &\le h_{\overline{D}}(y_n) + \frac{1}{n}h_{\overline{E}}(y_n) - \mu^{\mathrm{ess}}(\overline{D}) \\ &= h_{\overline{D}+\frac{1}{n}\overline{E}}(y_n) - \mu^{\mathrm{ess}}(\overline{D}) \le \mu^{\mathrm{ess}}\Big(\overline{D}+\frac{1}{n}\overline{E}\Big) - \mu^{\mathrm{ess}}(\overline{D}) + \frac{1}{n^2},\end{aligned}$$

which implies

$$\limsup_{n\to\infty} h_{\overline{E}}(y_n) \le \lim_{n\to\infty} n\left(\mu^{\mathrm{ess}}(\overline{D}+\frac{1}{n}\overline{E}) - \mu^{\mathrm{ess}}(\overline{D})\right) = \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}).$$

We conclude that $\lim_{n\to\infty} h_{\overline{E}}(y_n) = \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$, as stated. □

Proposition 4.3. *Let $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. The following conditions are equivalent:*

(1) for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ the limit $\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell)$ exists,

(2) $\partial_{-\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) = -\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$.

If they are satisfied, then $\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$ for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$.

Proof. The implication (2) ⇒ (1) is given by the first part of Lemma 4.2. Conversely, suppose that (1) holds. Then $\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$ for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$. Indeed, if the limit were different then using the second part of Lemma 4.2 we could construct a $\overline{D}$-small generic sequence $(x'_\ell)_\ell$ such that $(h_{\overline{E}}(x'_\ell))_\ell$ does not converge. Moreover, applying the latter to $-\overline{E}$ gives that every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ verifies

$$\partial_{-\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) = \lim_{\ell\to\infty} h_{-\overline{E}}(x_\ell) = -\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = -\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}),$$

which gives (2). □

The next result summarizes the relation between limit heights for $\overline{D}$-small sequences of algebraic points and the differentiability of the essential minimum function. It is a direct consequence of the previous one together with Lemma 3.2(2).

Proposition 4.4. *The following conditions are equivalent:*

(1) for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ and $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ the limit $\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell)$ exists,

(2) the essential minimum function is differentiable at $\overline{D}$.

If they are satisfied, then for any $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ we have

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) \quad \textit{for all } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

Let $v \in \mathfrak{M}_K$. For $x \in X(\overline{K})$ we denote by $\delta_{O(x)_v}$ the uniform probability measure on the v-adic Galois orbit of this point, that is

$$\delta_{O(x)_v} = \frac{1}{\# O(x)_v} \sum_{y \in O(x)_v} \delta_y.$$

Definition 4.5. We say that $\overline{D}$ satisfies the *equidistribution property* at v if there is a probability measure $\nu_{\overline{D},v}$ on X_v^{an} such that for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$, the sequence of measures $(\delta_{O(x_\ell)_v})_\ell$ converges to $\nu_{\overline{D},v}$. When this holds, $\nu_{\overline{D},v}$ is called the *(v-adic) equidistribution measure* of $\overline{D}$.

Note that if $\overline{D}$ satisfies the equidistribution property at a place v then its equidistribution measure $\nu_{\overline{D},v}$ is G_v-invariant, being the limit of a sequence of G_v-invariant discrete probability measures.

The next result gives the relation between the equidistribution property at v and the differentiability of the essential minimum function along the subspace of G_v-invariant continuous real-valued functions on X_v^{an}.

Proposition 4.6. *The following conditions are equivalent:*

(1) $\overline{D}$ satisfies the equidistribution property at v,

(2) the essential minimum function is differentiable at $\overline{D}$ along $C(X_v^{\mathrm{an}})^{G_v}$.

If they are satisfied, then the equidistribution measure $\nu_{\overline{D},v}$ is the unique G_v-invariant measure on X_v^{an} such that

$$n_v \int_{X_v^{\mathrm{an}}} \varphi \, d\nu_{\overline{D},v} = \partial_{\overline{0}^{\varphi}} \, \mu^{\mathrm{ess}}(\overline{D}) \quad \textit{for all } \varphi \in C(X_v^{\mathrm{an}})^{G_v}.$$

Proof. For every $\varphi \in C(X_v^{\mathrm{an}})^{G_v}$ and $x \in X(\overline{K})$ we have

$$h_{\overline{0}^{\varphi}}(x) = \frac{n_v}{\# O(x)_v} \sum_{y \in O(x)_v} \varphi(y) = n_v \int_{X_v^{\mathrm{an}}} \varphi \, d\delta_{O(x)_v}.$$

The statement then follows from Proposition 4.3 with Lemmas 3.2(2) and 2.20. □

4.2. **Main theorem.** Recall that $\overline{D}$ is an adelic $\mathbb{R}$-divisor on X with D big.

Definition 4.7. A *semipositive approximation* of $\overline{D}$ is a pair $(\phi, \overline{Q})$ where

(1) $\phi \colon X' \to X$ is a normal modification,

(2) $\overline{Q}$ is a semipositive adelic $\mathbb{R}$-divisor on X' with big geometric $\mathbb{R}$-divisor Q,

(3) $\phi^*\overline{D} - \overline{Q}$ is pseudo-effective.

When ϕ is the identity on X, we simply denote by $\overline{Q}$ the semipositive approximation of $\overline{D}$ corresponding to the pair $(\mathrm{Id}_X, \overline{Q})$.

The following is the central result of this text.

Theorem 4.8. *Assume that there exists a sequence $(\phi_n, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ such that*

$$\lim_{n \to \infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; \phi_n^* D)} = 0. \tag{4.2}$$

Then the essential minimum function is differentiable at $\overline{D}$ and

$$\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) = \lim_{n\to\infty} \frac{(\overline{Q}_n^d \cdot \phi_n^*\overline{E}) - d\,\mu^{\mathrm{ess}}(\overline{D})\,(Q_n^{d-1}\cdot \phi_n^* E)}{(Q_n^d)} \quad \textit{for all } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

In particular, the limit on the right-hand side exists in $\mathbb{R}$ and does not depend on the choice of the sequence.

This result together with Proposition 4.4 show that if $\overline{D}$ satisfies the condition (4.2), then for any $\overline{D}$-small generic sequence $(x_\ell)_\ell$ and $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ we have

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \lim_{n\to\infty} \frac{(\overline{Q}_n^d \cdot \phi_n^*\overline{E}) - d\,\mu^{\mathrm{ess}}(\overline{D})\,(Q_n^{d-1}\cdot \phi_n^* E)}{(Q_n^d)}.$$

In particular, $\overline{D}$ satisfies the equidistribution property at each place $v \in \mathfrak{M}_K$ and its v-adic equidistribution measure is given by

$$\nu_{\overline{D},v} = \lim_{n\to\infty} \nu_{n,v}$$

where $\nu_{n,v}$ denotes the pushforward to X_v^{an} of the normalized Monge-Ampère measure $c_1(\overline{Q}_{n,v})^{\wedge d}/(Q_n^d)$, as stated in Theorem 2.

Remark 4.9. The inradius $r(Q_n; \phi_n^* D)$ in Theorem 4.8 measures the bigness of the geometric $\mathbb{R}$-divisor Q_n for each n. For our purposes this invariant is finer than the geometric volume $\mathrm{vol}(Q_n) = (Q_n^d)$. Indeed, for any ample $A \in \mathrm{Div}(X)_{\mathbb{R}}$ with $A - D$ pseudo-effective we have that $\phi_n^* A - Q_n$ is pseudo-effective for all n, and so $r(Q_n; \phi_n^* D) \geq (Q_n^d)/(d\,(A^d))$ by Lemma 1.4. Hence any sequence of semipositive approximations $(\phi_n, \overline{Q}_n)_n$ of $\overline{D}$ such that

$$\lim_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{(Q_n^d)} = 0$$

also satisfies (4.2). When X is a curve this condition is equivalent to (4.2) but is stronger in higher dimensions, as it can be seen for instance in the semiabelian setting (Remark 7.12).

Remark 4.10. We assume throughout that X is normal because we do not consider adelic $\mathbb{R}$-divisors on an arbitrary projective variety. Nevertheless, Theorem 4.8 can be applied to study the equidistribution properties of an adelic divisor $\overline{D}$ on an arbitrary projective variety X over K. Indeed, since all the data is invariant under birational modifications one can reduce to the normal case by working on the normalization.

4.3. **Application to classical equidistribution results.** Yuan's equidistribution theorem (Theorem 1) is a direct consequence of Theorem 4.8, which moreover shows that this result is valid for any adelic $\mathbb{R}$-divisor with big geometric $\mathbb{R}$-divisor and gives the differentiability of the essential minimum function. These slight improvements could already be obtained by adapting Yuan's proof.

Corollary 4.11. *Assume that $\overline{D}$ is semipositive and that*

$$\mu^{\mathrm{ess}}(\overline{D}) = \frac{(\overline{D}^{d+1})}{(d+1)\,(D^d)}.$$

Then the essential minimum function is differentiable at $\overline{D}$ and

$$\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) = \frac{(\overline{D}^d \cdot \overline{E}) - d\,\mu^{\mathrm{ess}}(\overline{D})\,(D^{d-1}\cdot E)}{(D^d)} \quad \textit{for all } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

In particular, $\overline{D}$ satisfies the v-adic equidistribution property at each $v \in \mathfrak{M}_K$ with equidistribution measure $\nu_{\overline{D},v} = c_1(\overline{D}_v)^{\wedge d}/(D^d)$.

Proof. Apply Theorem 4.8 to the sequence of semipositive approximations $\overline{Q}_n = \overline{D}$, $n \in \mathbb{N}$. By Theorem 2.19 we have $\mu^{\mathrm{abs}}(\overline{D}) = \mu^{\mathrm{ess}}(\overline{D})$ and so (4.2) is verified. □

The following corollary recovers Chen's equidistribution theorem [Che11, Corollary 5.5]. As shown in *loc. cit.*, this result is a consequence of the differentiability of the arithmetic volume. We shall explain how to deduce it from Theorem 4.8.

Corollary 4.12. *Assume that $\overline{D}$ is big and that*

$$\mu^{\mathrm{ess}}(\overline{D}) = \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\,\mathrm{vol}(D)}.$$

Then the essential minimum function is differentiable at $\overline{D}$ and

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \frac{(\langle \overline{D}^d \rangle \cdot \overline{E}) - d\, \mu^{\mathrm{ess}}(\overline{D})\, (\langle D^{d-1} \rangle \cdot E)}{\mathrm{vol}(D)} \qquad \text{for all } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

In particular, $\overline{D}$ satisfies the v-adic equidistribution property at each $v \in \mathfrak{M}_K$, with equidistribution measure $\nu_{\overline{D},v} = \omega_{\overline{D},v}$.

Here $\omega_{\overline{D},v}$ is the probability measure on X_v^{an} in (3.5) and $(\langle D^{d-1} \rangle \cdot E)$ is the geometric positive intersection number from [BFJ09] (see Remark 3.9). We need the next lemma to construct a suitable sequence of semipositive approximations of $\overline{D}$.

Lemma 4.13. *For every real numbers $t < \mu^{\mathrm{ess}}(\overline{D})$ and $\varepsilon > 0$ there exists a semipositive approximation $(\phi, \overline{Q})$ of $\overline{D}$ such that*

$$\mu^{\mathrm{abs}}(\overline{Q}) \geq t, \quad |(Q^d) - \mathrm{vol}(R^t(\overline{D}))| \leq \varepsilon, \quad |(\overline{Q}^{d+1}) - \widehat{\mathrm{vol}}(\overline{D}(t)) - (d+1)t\, \mathrm{vol}(R^t(\overline{D}))| \leq \varepsilon.$$

Proof. By Theorem 2.17, $\overline{D}(t)$ is big. Then by Proposition 3.6 applied to $\overline{D}(t)$, for any $\varepsilon' > 0$ there is a nef adelic $\mathbb{R}$-divisor $\overline{P}$ on a normal modification $\phi\colon X' \to X$ with $\phi^*\overline{D}(t) - \overline{P}$ pseudo-effective such that

$$|(P^d) - \mathrm{vol}(R^t(\overline{D}))| \leq \varepsilon' \quad \text{and} \quad |(\overline{P}^{d+1}) - \widehat{\mathrm{vol}}(\overline{D}(t))| \leq \varepsilon'. \tag{4.3}$$

Set $\overline{Q} = \overline{P}(-t) = \overline{P} + t[\infty]$. By construction, $(\phi, \overline{Q})$ is a semipositive approximation of $\overline{D}$. Since $\overline{P}$ is nef we have $\mu^{\mathrm{abs}}(\overline{Q}) = \mu^{\mathrm{abs}}(\overline{P}) + t \geq t$, which gives the first condition. By (2.4) we also have

$$(Q^d) = (P^d) \quad \text{and} \quad (\overline{Q}^{d+1}) = (\overline{P}^{d+1}) + (d+1)\, t\, (Q^d),$$

and so the second and third conditions follow from (4.3). □

Proof of Corollary 4.12. By Theorem 2.8 and Lemma 2.16(3) we have

$$\mu^{\mathrm{ess}}(\overline{D}) = \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\,\mathrm{vol}(D)} = \frac{1}{\mathrm{vol}(D)} \int_0^{\mu^{\mathrm{ess}}(\overline{D})} \mathrm{vol}(R^t(\overline{D}))\, dt.$$

Since $\mathrm{vol}(R^t(\overline{D})) \leq \mathrm{vol}(D)$ we get $\mathrm{vol}(R^t(\overline{D})) = \mathrm{vol}(D)$ for all $0 \leq t \leq \mu^{\mathrm{ess}}(\overline{D})$. Hence by the same results, for t in this range we have

$$\widehat{\mathrm{vol}}(\overline{D}(t)) = \widehat{\mathrm{vol}}(\overline{D}) - (d+1)\, t\, \mathrm{vol}(D).$$

Applying Lemma 4.13 we deduce that there is a sequence $(\phi_n, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ such that

$$\lim_{n\to\infty} (\overline{Q}_n^{d+1}) = \widehat{\mathrm{vol}}(\overline{D}), \quad \lim_{n\to\infty} (Q_n^d) = \mathrm{vol}(D), \quad \lim_{n\to\infty} \mu^{\mathrm{abs}}(\overline{Q}_n) = \mu^{\mathrm{ess}}(\overline{D}).$$

Hence by Remark 4.9 the sequence $(\phi_n, \overline{Q}_n)_n$ satisfies the condition (4.2). Since $\mu^{\mathrm{ess}}(\overline{D}) > 0$, we have that $\overline{Q}_n$ is nef for sufficiently large n and therefore $(\phi_n, \overline{Q}_n)_n$ is a Fujita approximation sequence of $\overline{D}$. By the definition of arithmetic positive intersection numbers and Remark 3.9 we have

$$\lim_{n\to\infty} (\overline{Q}_n^d \cdot \phi_n^* \overline{E}) = (\langle \overline{D}^d \rangle \cdot \overline{E}) \quad \text{and} \quad \lim_{n\to\infty} (Q_n^{d-1} \cdot \phi_n^* E) = (\langle D^{d-1} \rangle \cdot E).$$

The result then follows from Theorem 4.8. □

Remark 4.14. Chen's equidistribution theorem implies Yuan's. Indeed, for $\overline{D}$ semipositive the statement of the latter is invariant under shifts of this adelic $\mathbb{R}$-divisor by multiples of $[\infty]$. Hence we can suppose without loss of generality that $\overline{D}$ is big and nef, in which case Corollary 4.12 specializes to Corollary 4.11.

4.4. **Interpretation in terms of arithmetic positive intersection numbers.** Here we propose a reformulation of Theorem 4.8 that gives a more intrinsic condition for the differentiability of the essential minimum function and shows that the derivative can be computed using limits of arithmetic positive intersection numbers. As before, we let $\overline{D}$ be an $\mathbb{R}$-divisor on X with D big.

We define the inradius of a big adelic $\mathbb{R}$-divisor as the supremum of the geometric inradii of its nef approximations.

Definition 4.15. Let $\overline{B}$ be a big adelic $\mathbb{R}$-divisor on X. A *nef approximation* of $\overline{B}$ is a semipositive approximation $(\phi, \overline{P})$ of $\overline{B}$ such that $\overline{P}$ is nef. We denote by $\Theta(\overline{B})$ the set of nef approximations of $\overline{B}$.

For a big $\mathbb{R}$-divisor A on X, the *inradius* of $\overline{B}$ with respect to A is defined as

$$\rho(\overline{B}; A) = \sup\{ r(P; \phi^* A) \mid (\phi, \overline{P}) \in \Theta(\overline{B}) \}.$$

This is a positive real number. We also set $\rho(\overline{B}) = \rho(\overline{B}; B)$ for the inradius of $\overline{B}$ with respect to its geometric $\mathbb{R}$-divisor B, which is big.

In the next result and similar ones, the limits when t tends to the essential minimum are taken *from below*. Note that for every real number $t < \mu^{\mathrm{ess}}(\overline{D})$ we have that $\overline{D}(t)$ is big by Theorem 2.17, and so the inradius $\rho(\overline{D}(t))$ is well-defined.

Theorem 4.16. *Assume that*

$$\liminf_{t\to\mu^{\mathrm{ess}}(\overline{D})} \frac{\mu^{\mathrm{ess}}(\overline{D}) - t}{\rho(\overline{D}(t))} = 0. \tag{4.4}$$

Then the essential minimum function is differentiable at $\overline{D}$ and

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \lim_{t\to\mu^{\mathrm{ess}}(\overline{D})} \frac{(\langle \overline{D}(t)^d \rangle \cdot \overline{E})}{\mathrm{vol}(R^t(\overline{D}))} \quad \textit{for all } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}. \tag{4.5}$$

We can reformulate accordingly the asymptotic behavior of heights and Galois orbits of the algebraic points in a $\overline{D}$-small generic sequence: by Proposition 4.3, this result shows that if $\overline{D}$ satisfies the condition (4.4) then for any $\overline{D}$-small generic sequence $(x_\ell)_\ell$ and $\overline{E} \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$ we have

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \lim_{t\to\mu^{\mathrm{ess}}(\overline{D})} \frac{(\langle \overline{D}(t)^d\rangle \cdot \overline{E})}{\mathrm{vol}(R^t(\overline{D}))}.$$

In particular, $\overline{D}$ satisfies the equidistribution property at each place $v \in \mathfrak{M}_K$ and its v-adic equidistribution measure is given by

$$\nu_{\overline{D},v} = \lim_{t\to\mu^{\mathrm{ess}}(\overline{D})} \omega_{\overline{D}(t),v},$$

where $\omega_{\overline{D}(t),v}$ is the probability measure on X_v^{an} from (3.5).

We first observe that the conditions in Theorems 4.8 and 4.16 are equivalent.

Lemma 4.17. *The condition* (4.4) *holds if and only if there exists a sequence* $(\phi_n, \overline{Q}_n)_n$ *of semipositive approximations of* $\overline{D}$ *satisfying the condition* (4.2)*.*

Proof. First assume that (4.4) holds. Then there are sequences of real numbers $t_n < \mu^{\mathrm{ess}}(\overline{D})$, $n \in \mathbb{N}$, and of nef approximations $(\phi_n, \overline{P}_n) \in \Theta(\overline{D}(t_n))$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - t_n}{r(P_n; \phi_n^* D)} = 0.$$

For each n put $\overline{Q}_n = \overline{P}_n(-t_n)$. Then $(\phi_n, \overline{Q}_n)$ is a semipositive approximation of $\overline{D}$ with $Q_n = P_n$ and $\mu^{\mathrm{abs}}(\overline{Q}_n) = \mu^{\mathrm{abs}}(\overline{P}_n) + t_n \geq t_n$. Therefore the sequence $(\phi_n, \overline{Q}_n)_n$ satisfies (4.2).

Conversely let $(\phi_n, \overline{Q}_n)_n$ be a sequence of semipositive approximations of $\overline{D}$ satisfying (4.2). For each n set $t_n = \mu^{\mathrm{abs}}(\overline{Q}_n)$ and $\overline{P}_n = \overline{Q}_n(t_n)$. Then $\overline{P}_n$ is nef since it is semipositive and $\mu^{\mathrm{abs}}(\overline{P}_n) = \mu^{\mathrm{abs}}(\overline{Q}_n) - t_n = 0$. Moreover

$$\phi_n^* \overline{D}(t_n) - \overline{P}_n = (\phi_n^* \overline{D} - t_n[\infty]) - (\overline{Q}_n - t_n[\infty]) = \phi_n^* \overline{D} - \overline{Q}_n$$

is pseudo-effective. Therefore $(\phi_n, \overline{P}_n) \in \Theta(\overline{D}(t_n))$ and in particular

$$\rho(\overline{D}(t_n)) \geq r(P_n; \phi_n^* D) = r(Q_n; \phi_n^* D).$$

Finally we have

$$0 \leq \liminf_{t\to\mu^{\mathrm{ess}}(\overline{D})} \frac{\mu^{\mathrm{ess}}(\overline{D}) - t}{\rho(\overline{D}(t))} \leq \liminf_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - t_n}{\rho(\overline{D}(t_n))} \leq \lim_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; \phi_n^* D)} = 0.$$

□

Note that for every $t < \mu^{\mathrm{ess}}(\overline{D})$ and $\overline{E} \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$ we have

$$\Omega_{\overline{D}(t)}(\overline{E}) = \frac{(\langle \overline{D}(t)^d\rangle \cdot \overline{E})}{\mathrm{vol}(R^t(\overline{D}))},$$

where $\Omega_{\overline{D}(t)} \colon \widehat{\mathrm{Div}}(X)_\mathbb{R} \to \mathbb{R}$ is the linear functional defined in (3.3).

Lemma 4.18. *For every* $\overline{E} \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$ *we have*

$$-\partial_{-\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \limsup_{t\to\mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E}) \geq \liminf_{t\to\mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E}) \geq \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}).$$

In particular, if the essential minimum function is differentiable at $\overline{D}$ *then the limit* $\lim_{t\to\mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E})$ *exists and equals* $\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$*.*

Proof. Let $t < \mu^{\mathrm{ess}}(\overline{D})$. As observed in Remark 3.10, $\Omega_{\overline{D}(t)}$ takes nonnegative values on pseudo-effective adelic $\mathbb{R}$-divisors and verifies $\Omega_{\overline{D}(t)}([\infty]) = 1$. By Theorem 2.17, for each $\lambda > 0$ we have that $\overline{D} + \lambda\overline{E} - \mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E})\,[\infty]$ is pseudo-effective and so

$$\Omega_{\overline{D}(t)}(\overline{D}) + \lambda\,\Omega_{\overline{D}(t)}(\overline{E}) = \Omega_{\overline{D}(t)}(\overline{D} + \lambda\,\overline{E}) \geq \mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E}). \tag{4.6}$$

On the other hand

$$\Omega_{\overline{D}(t)}(\overline{D}) = \Omega_{\overline{D}(t)}(\overline{D}(t)) + t = \frac{\widehat{\mathrm{vol}}(\overline{D}(t))}{\mathrm{vol}(R^t(\overline{D}))} + t \leq (d+1)(\mu^{\mathrm{ess}}(\overline{D}) - t) + t$$

by the first formula in (3.2) and Zhang's inequality (Theorem 2.18). Therefore

$$\lim_{t \to \mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{D}) = \mu^{\mathrm{ess}}(\overline{D}).$$

Taking the infimum limit as t approaches $\mu^{\mathrm{ess}}(\overline{D})$ from below in (4.6) then gives

$$\liminf_{t \to \mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E}) \geq \frac{\mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E}) - \mu^{\mathrm{ess}}(\overline{D})}{\lambda},$$

and we obtain $\liminf_{t \to \mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E}) \geq \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$ by letting $\lambda \to 0$. The rest of the statement follows by applying this to $-\overline{E}$. □

Proof of Theorem 4.16. If (4.4) holds, then by Lemma 4.17 and Theorem 4.8 the essential minimum function is differentiable at $\overline{D}$. The expression for the derivative in (4.5) is given by Lemma 4.18. □

It would be very interesting to determine whether the condition in Theorem 4.16 is actually a criterion for the differentiability of the essential minimum.

Question 4.19. If the essential minimum function is differentiable at $\overline{D}$, then does

$$\liminf_{t \to \mu^{\mathrm{ess}}(\overline{D})} \frac{\mu^{\mathrm{ess}}(\overline{D}) - t}{\rho(\overline{D}(t))} = 0$$

necessarily hold? More optimistically, does this hold as soon as $\overline{D}$ has the equidistribution property at every place?

In Section 5.5 we give a partial answer to this question.

5. Proof of Theorem 4.8 and complements

In this section we prove our main result and give some complements, including an equidistribution theorem with a more flexible condition for the sequence of semipositive approximations, a logarithmic equidistribution theorem, and a partial converse to Theorem 4.8.

5.1. **A consequence of the arithmetic Siu's inequality.** The following result is a variant of Lemma 3.5 with an error term depending on an inradius.

Proposition 5.1. *Let $\overline{P}, \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with P big and $\overline{P}$ nef. Assume that there exists a nef adelic $\mathbb{R}$-divisor $\overline{A}$ on X such that A is big, $\overline{A} + \overline{E}$ is pseudo-effective and $\overline{A} - \overline{E}$ is nef. There is a constant c_d depending only on d such that for every $\lambda \geq 0$*

$$\widehat{\mathrm{vol}}_{\chi}(\overline{P} + \lambda\overline{E}) \geq (\overline{P}^{d+1}) + (d+1)\,(\overline{P}^{d} \cdot \overline{E})\,\lambda - c_d \max\Big(1, \Big(\frac{\lambda}{r(P; A)}\Big)^{d-1}\Big)\,\frac{(\overline{P}^{d} \cdot \overline{A})}{r(P; A)}\,\lambda^2.$$

Its proof combines Yuan's arithmetic analogue of Siu's inequality (Theorem 2.11) with the following lemma, itself a consequence of the arithmetic Hodge index theorem due to Yuan and Zhang [YZ17]. The first point is proven along the lines of [Iko15, Theorem 2.7(2)].

Lemma 5.2. *Let $\overline{P}$ and $\overline{A}$ be nef adelic $\mathbb{R}$-divisors on X with P, A big. Then*

(1) for $i = 1, \dots, d$ we have $(P^{d+1-i} \cdot A^{i-1})\,(\overline{P}^{d-i} \cdot \overline{A}^{i+1}) \leq 2\,(P^{d-i} \cdot A^{i})\,(\overline{P}^{d+1-i} \cdot \overline{A}^{i})$,

(2) for $i = 0, \dots, d$ we have $(\overline{P}^{d-i} \cdot \overline{A}^{i+1}) \leq \Big(\dfrac{2}{r(P;A)}\Big)^{i}\,(\overline{P}^{d} \cdot \overline{A})$.

Proof. As P and A are big and nef we have $(P^d) = \mathrm{vol}(P) > 0$ and $(A^d) = \mathrm{vol}(A) > 0$, which implies that $(P^{d+1-i} \cdot A^{i-1}) > 0$ by [Laz04, Theorem 1.6.1]. Set

$$\alpha = \frac{(P^{d-i} \cdot A^{i})}{(P^{d+1-i} \cdot A^{i-1})},$$

so that $((A - \alpha P) \cdot P^{d-i} \cdot A^{i-1}) = 0$. By the arithmetic Hodge index theorem [YZ17, Theorem 2.2] (which remains valid for adelic $\mathbb{R}$-divisors by [Iko15, Theorem 2.7(1)]) this implies

$$((\overline{A} - \alpha\overline{P})^2 \cdot \overline{P}^{d-i} \cdot \overline{A}^{i-1}) \leq 0. \tag{5.1}$$

On the other hand

$$((\overline{A} - \alpha\overline{P})^2 \cdot \overline{P}^{d-i} \cdot \overline{A}^{i-1}) = (\overline{P}^{d-i} \cdot \overline{A}^{i+1}) - 2\alpha\,(\overline{P}^{d+1-i} \cdot \overline{A}^{i}) + \alpha^2\,(\overline{P}^{d+2-i} \cdot \overline{A}^{i-1})$$

and $(\overline{P}^{d+2-i} \cdot \overline{A}^{i-1}) \geq 0$ since both $\overline{P}$ and $\overline{A}$ are nef. Therefore (1) follows from (5.1).

Note that (2) holds trivially for $i = 0$. We deduce the general case by induction on i, applying (1) and the inequality

$$(P^{d-i} \cdot A^{i}) \leq \frac{1}{r(P;A)}(P^{d+1-i} \cdot A^{i-1}),$$

which follows from (1.1) and the fact that $P - r(P;A)A$ is pseudo-effective. □

Proof of Proposition 5.1. Set $\overline{B} = \overline{A} - \overline{E}$. This is a nef adelic $\mathbb{R}$-divisor on X and so by Theorem 2.11 we have

$$\widehat{\mathrm{vol}}_{\chi}(\overline{P} + \lambda\overline{E}) = \widehat{\mathrm{vol}}_{\chi}(\overline{P} + \lambda\overline{A} - \lambda\overline{B}) \geq ((\overline{P} + \lambda\overline{A})^{d+1}) - (d+1)\,((\overline{P} + \lambda\overline{A})^{d} \cdot \overline{B})\,\lambda.$$

Expanding the right-hand side we find that it is equal to

$$(\overline{P}^{d+1}) + (d+1)\,(\overline{P}^{d} \cdot \overline{E})\,\lambda + \sum_{i=2}^{d+1} \binom{d+1}{i} (\overline{P}^{d+1-i} \cdot \overline{A}^{i})\,\lambda^{i}$$
$$- (d+1) \sum_{i=1}^{d} \binom{d}{i} (\overline{P}^{d-i} \cdot \overline{A}^{i} \cdot \overline{B})\,\lambda^{i+1}.$$

Since $\overline{P}$ and $\overline{A}$ are nef, the first sum is nonnegative and therefore

$$\widehat{\mathrm{vol}}_{\chi}(\overline{P} + \lambda\overline{E}) \geq (\overline{P}^{d+1}) + (d+1)\,(\overline{P}^{d} \cdot \overline{E})\,\lambda - (d+1) \sum_{i=1}^{d} \binom{d}{i} (\overline{P}^{d-i} \cdot \overline{A}^{i} \cdot \overline{B})\,\lambda^{i+1}.$$

By Lemma 2.13 and the fact that $2\overline{A}-\overline{B}=\overline{A}+\overline{E}$ is pseudo-effective we have $(\overline{P}^{d-i}\cdot\overline{A}^{i}\cdot\overline{B})\le 2\,(\overline{P}^{d-i}\cdot\overline{A}^{i+1})$, $i=1,\dots,d$, and by Lemma 5.2(2),

$$(\overline{P}^{d-i}\cdot\overline{A}^{i+1})\le\Big(\frac{2}{r(P;A)}\Big)^{i}(\overline{P}^{d}\cdot\overline{A}).$$

Therefore $\widehat{\mathrm{vol}}_{\chi}(\overline{P}+\lambda\overline{E})$ is bounded from below by

$$(\overline{P}^{d+1})+(d+1)\,(\overline{P}^{d}\cdot\overline{E})\,\lambda-(d+1)\,(\overline{P}^{d}\cdot\overline{A})\sum_{i=1}^{d}\binom{d}{i}\frac{2^{i+1}\lambda^{i+1}}{r(P;A)^{i}},$$

and the result follows. □

The following consequence of Proposition 5.1 plays a central role in our proof.

Corollary 5.3. *Let $\overline{P},\overline{E}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with $\overline{P}$ nef and P big. Assume that there exists a nef adelic $\mathbb{R}$-divisor $\overline{A}$ such that A is big, $\overline{A}+\overline{E}$ is pseudo-effective and $\overline{A}-\overline{E}$ is nef. There exists a constant c_d depending only on d such that*

$$\mu^{\mathrm{ess}}(\overline{P}+\lambda\overline{E})\ge\frac{(\overline{P}^{d+1})}{(d+1)\,\mathrm{vol}(P+\lambda E)}+\frac{(\overline{P}^{d}\cdot\overline{E})}{(P^{d})}\,\lambda-c_d\,\frac{(\overline{P}^{d}\cdot\overline{A})}{(P^{d})}\,\frac{\lambda^{2}}{r(P;A)}$$

for every $0\le\lambda<r(P;A)/2$. In particular, if $E=0$ then

$$\mu^{\mathrm{ess}}(\overline{P}+\lambda\overline{E})\ge\frac{(\overline{P}^{d+1})}{(d+1)(P^{d})}+\frac{(\overline{P}^{d}\cdot\overline{E})}{(P^{d})}\,\lambda-c_d\,\frac{(\overline{P}^{d}\cdot\overline{A})}{(P^{d})}\,\frac{\lambda^{2}}{r(P;A)}.$$

Proof. Let λ be a real number with $0\le\lambda<r(P;A)/2$. By Lemma 1.5 we have

$$\Big(1-\frac{\lambda}{r(P;A)}\Big)^{d}(P^{d})\le\mathrm{vol}(P+\lambda E)\le\Big(1+\frac{\lambda}{r(P;A)}\Big)^{d}(P^{d}).\tag{5.2}$$

In particular, $P+\lambda E$ is big. By Zhang's inequality (Theorem 2.18) we have

$$\mu^{\mathrm{ess}}(\overline{P}+\lambda\overline{E})\ge\frac{\widehat{\mathrm{vol}}_{\chi}(\overline{P}+\lambda\overline{E})}{(d+1)\,\mathrm{vol}(P+\lambda E)}.$$

Therefore Proposition 5.1 implies that

$$\mu^{\mathrm{ess}}(\overline{P}+\lambda\overline{E})\ge\frac{(\overline{P}^{d+1})}{(d+1)\,\mathrm{vol}(P+\lambda E)}+\frac{(\overline{P}^{d}\cdot\overline{E})}{\mathrm{vol}(P+\lambda E)}\,\lambda-c_d\,\frac{(\overline{P}^{d}\cdot\overline{A})}{\mathrm{vol}(P+\lambda E)}\,\frac{\lambda^{2}}{r(P;A)}\tag{5.3}$$

for a constant $c_d>0$ depending only on d. By (5.2) we have

$$\frac{(\overline{P}^{d}\cdot\overline{A})}{\mathrm{vol}(P+\lambda E)}\le\Big(1-\frac{\lambda}{r(P;A)}\Big)^{-d}\frac{(\overline{P}^{d}\cdot\overline{A})}{(P^{d})}\le 2^{d}\,\frac{(\overline{P}^{d}\cdot\overline{A})}{(P^{d})}.$$

On the other hand, by Lemma 2.13 we have $|(\overline{P}^{d}\cdot\overline{E})|\le(\overline{P}^{d}\cdot\overline{A})$ since $\overline{P}$ is nef and $\overline{A}\pm\overline{E}$ are pseudo-effective. Set $a=1$ if $(\overline{P}^{d}\cdot\overline{E})\ge 0$ and $a=-1$ otherwise. Then by (5.2) we have

$$\begin{aligned}\frac{(\overline{P}^{d}\cdot\overline{E})}{\mathrm{vol}(P+\lambda E)}&\ge\frac{(\overline{P}^{d}\cdot\overline{E})}{(P^{d})}\Big(1+a\,\frac{\lambda}{r(P;A)}\Big)^{-d}\\&\ge\frac{(\overline{P}^{d}\cdot\overline{E})}{(P^{d})}-c_d'\,\frac{|(\overline{P}^{d}\cdot\overline{E})|}{(P^{d})}\,\frac{\lambda}{r(P;A)}\ge\frac{(\overline{P}^{d}\cdot\overline{E})}{(P^{d})}-c_d'\,\frac{(\overline{P}^{d}\cdot\overline{A})}{(P^{d})}\,\frac{\lambda}{r(P;A)}\end{aligned}$$

for some constant c'_d depending only on d. The result follows by combining these inequalities with (5.3). $\square$

5.2. **Proof of Theorem 4.8.** Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with D big and $(\phi_n, \overline{Q}_n)_n$ a sequence of semipositive approximations of $\overline{D}$ satisfying the condition (4.2).

Lemma 5.4. *For every big $\mathbb{R}$-divisor A on X we have* $\displaystyle\lim_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; \phi_n^* A)} = 0.$

Proof. This a straightforward consequence of (4.2) thanks to Lemma 1.2. $\square$

For each $n \in \mathbb{N}$ set

$$\widetilde{Q}_n = \overline{Q}_n - \mu^{\mathrm{abs}}(\overline{Q}_n)\, [\infty] \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

Note that $\mu^{\mathrm{abs}}(\widetilde{Q}_n) = \mu^{\mathrm{abs}}(\overline{Q}_n) - \mu^{\mathrm{abs}}(\overline{Q}_n) = 0$ and therefore $\widetilde{Q}_n$ is nef.

Lemma 5.5. *Let $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. We have*

$$\frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{E})}{(Q_n^d)} = \frac{(\overline{Q}_n^d \cdot \phi_n^* \overline{E}) - d\, \mu^{\mathrm{abs}}(\overline{Q}_n)\, (Q_n^{d-1} \cdot \phi_n^* E)}{(Q_n^d)} \quad \textit{for all } n \in \mathbb{N}$$

and $$\lim_{n\to\infty} \Big(\frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{E})}{(Q_n^d)} - \frac{(\overline{Q}_n^d \cdot \phi_n^* \overline{E}) - d\, \mu^{\mathrm{ess}}(\overline{D})\, (Q_n^{d-1} \cdot \phi_n^* E)}{(Q_n^d)} \Big) = 0.$$

Proof. The first equality follows from the multilinearity of the arithmetic intersection product and the formula (2.4). For the second set

$$\beta_n = (\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n))\, \frac{(Q_n^{d-1} \cdot \phi_n^* E)}{(Q_n^d)}, \quad n \in \mathbb{N},$$

so that this statement is equivalent to $\lim_{n\to\infty} \beta_n = 0$. To see this, let A be an ample divisor such that $A \pm E$ are big. We have $|(Q_n^d \cdot \phi_n^* E)| \le (Q_n^d \cdot \phi_n^* A)$ by the inequality (1.1), and using Lemma 1.3 we get

$$|\beta_n| \le (\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n))\, \frac{(Q_n^{d-1} \cdot \phi_n^* A)}{(Q_n^d)} \le \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; \phi_n^* A)}.$$

We conclude with Lemma 5.4. $\square$

Lemma 5.6. *For every nef $\overline{A} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with A big we have* $\displaystyle\sup_{n\in\mathbb{N}} \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{A})}{(Q_n^d)} < \infty.$

Proof. Up to replacing A by εA for a small $\varepsilon > 0$ we can assume that $D - A$ is big. Then up to replacing $\overline{A}$ by $\overline{A} - c\,[\infty]$ for $c \in \mathbb{R}$ sufficiently large we can furthermore assume that $\overline{D} - \overline{A}$ is big thanks to Lemma 2.7.

For every $n \in \mathbb{N}$ and $\lambda > 0$ we have

$$\lambda\, (\widetilde{Q}_n^d \cdot \phi_n^* \overline{A}) \le \frac{1}{d+1} ((\widetilde{Q}_n + \lambda\, \phi_n^* \overline{A})^{d+1}) \le ((Q_n + \lambda\, \phi_n^* A)^d)\, \mu^{\mathrm{ess}}(\widetilde{Q}_n + \lambda\, \phi_n^* \overline{A}),$$

where the first inequality follows from the fact that both $\widetilde{Q}_n$ and $\overline{A}$ are nef, and the second from Zhang's inequality (Theorem 2.18). By Lemma 2.16(4) and the fact that $\phi_n^* \overline{D} - \overline{Q}_n$ and $\overline{D} - \overline{A}$ are pseudo-effective we have

$$\mu^{\mathrm{ess}}(\widetilde{Q}_n + \lambda\, \phi_n^* \overline{A}) = \mu^{\mathrm{ess}}(\overline{Q}_n + \lambda\, \phi_n^* \overline{A}) - \mu^{\mathrm{abs}}(\overline{Q}_n) \le (1+\lambda)\, \mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n).$$

Set $r_n = r(Q_n; \phi_n^* A)$ for short. Since $Q_n - r_n\, \phi_n^* A$ is pseudo-effective we also have

$$((Q_n + r_n \phi_n^* A)^d) = \mathrm{vol}(Q_n + r_n \phi_n^* A) \le \mathrm{vol}(2\, Q_n) = 2^d (Q_n^d).$$

Combining the previous inequalities for $\lambda = r_n$ we get

$$r_n\,(\widetilde{Q}_n^d \cdot \phi_n^*\overline{A}) \leq 2^d\,((1+r_n)\,\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n))\,(Q_n^d).$$

Hence

$$\frac{(\widetilde{Q}_n^d \cdot \phi_n^*\overline{A})}{(Q_n^d)} \leq 2^d\,\frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r_n} + 2^d\mu^{\mathrm{ess}}(\overline{D}).$$

By Lemma 5.4 the right-hand side is upper-bounded by some constant independent of n, and the result follows. □

Proof of Theorem 4.8. Recall from Section 4.1 that for every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, the one-sided derivative $\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$ exists and $-\partial_{-\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) \geq \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$. We claim that

$$\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) \geq \limsup_{n\to\infty} \frac{(\widetilde{Q}_n^d \cdot \phi_n^*\overline{E})}{(Q_n^d)}. \tag{5.4}$$

We first prove this when $\overline{E}$ is DSP. In that case, by Lemma 2.14 there exists a big and nef $\overline{A} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ such that $\overline{A} \pm \overline{E}$ are nef. By Lemma 5.6,

$$\kappa := \sup_{n\in\mathbb{N}} \frac{(\widetilde{Q}_n^d \cdot \phi_n^*\overline{A})}{(Q_n^d)}$$

is a real number. For each $n \in \mathbb{N}$ and any $\lambda \geq 0$ such that $D + \lambda E$ is big we have

$$\mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E}) - \mu^{\mathrm{abs}}(\overline{Q}_n) \geq \mu^{\mathrm{ess}}(\overline{Q}_n + \lambda\,\phi_n^*\overline{E}) - \mu^{\mathrm{abs}}(\overline{Q}_n) = \mu^{\mathrm{ess}}(\widetilde{Q}_n + \lambda\,\phi_n^*\overline{E})$$

by Lemma 2.16(4). Since $\widetilde{Q}_n$ is nef we have $(\widetilde{Q}_n)^{d+1} \geq 0$, and so by Corollary 5.3 applied to $\overline{P} = \widetilde{Q}_n$ we get

$$\mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E}) - \mu^{\mathrm{abs}}(\overline{Q}_n) \geq \frac{(\widetilde{Q}_n^d \cdot \phi_n^*\overline{E})}{(Q_n^d)}\,\lambda - \frac{c_d\,\kappa}{r(Q_n;\phi_n^*A)}\,\lambda^2$$

for every $0 \leq \lambda < r(Q_n;\phi_n^*A)/2$, where c_d is a constant depending only on d. Rearranging this we obtain

$$\frac{\mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E}) - \mu^{\mathrm{ess}}(\overline{D})}{\lambda} \geq \frac{(\widetilde{Q}_n^d \cdot \phi_n^*\overline{E})}{(Q_n^d)} - \frac{\mu^{\mathrm{ess}}(\overline{D})) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{\lambda} - \frac{c_d\,\kappa}{r(Q_n;\phi_n^*A)}\,\lambda. \tag{5.5}$$

Set

$$\gamma_n = \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n;\phi_n^*A)} \quad \text{if } \mu^{\mathrm{abs}}(\overline{Q}_n) \neq \mu^{\mathrm{ess}}(\overline{D}) \quad \text{and} \quad \gamma_n = \frac{1}{n} \quad \text{otherwise},$$

and then $\lambda_n = r(Q_n;\phi_n^*A)\,\gamma_n^{1/2}$. By Lemma 5.4 we have $\lim_{n\to\infty}\gamma_n = 0$ and so $\lim_{n\to\infty}\lambda_n = 0$. Applying (5.5) with $\lambda = \lambda_n$ gives

$$\frac{\mu^{\mathrm{ess}}(\overline{D} + \lambda_n\overline{E}) - \mu^{\mathrm{ess}}(\overline{D})}{\lambda_n} \geq \frac{(\widetilde{Q}_n^d \cdot \phi_n^*\overline{E})}{(Q_n^d)} - \gamma_n^{1/2} - c_d\,\kappa\,\gamma_n^{1/2},$$

and we obtain (5.4) by letting $n \to \infty$.

We now consider the general case. By Lemma 2.12, for each $\varepsilon > 0$ there is a DSP $\overline{E}' \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with $\overline{E} - \overline{E}'$ and $\overline{E}' - \overline{E} + \varepsilon[\infty]$ pseudo-effective. Then $\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) \geq \partial_{\overline{E}'}\,\mu^{\mathrm{ess}}(\overline{D})$ by Lemma 2.16(4), and Lemma 2.13 together with the formula (2.4) gives

$$(\widetilde{Q}_n^d \cdot \overline{E}') \geq (\widetilde{Q}_n^d \cdot (\overline{E} - \varepsilon\,[\infty])) = (\widetilde{Q}_n^d \cdot \overline{E}) - \varepsilon\,(Q_n^d)$$

for all $n \in \mathbb{N}$. By the DSP case we obtain

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \partial_{\overline{E}'}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \limsup_{n\to\infty} \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{E}')}{(Q_n^d)} \geq \limsup_{n\to\infty} \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{E})}{(Q_n^d)} - \varepsilon,$$

and we conclude by letting $\varepsilon \to 0$.

Finally, applying (5.4) to $-\overline{E}$ and $\overline{E}$ we get

$$\liminf_{n\to\infty} \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{E})}{(Q_n^d)} \geq -\partial_{-\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \limsup_{n\to\infty} \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{E})}{(Q_n^d)}.$$

Hence $-\partial_{-\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$, and we conclude with Lemmas 3.2 and 5.5. □

Remark 5.7. In the setting of Theorem 4.8, it follows from Lemma 5.5 that if the condition (4.2) is satisfied then the derivatives of the essential minimum function at $\overline{D}$ can be alternatively expressed as

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \lim_{n\to\infty} \frac{(\overline{Q}_n^d \cdot \phi_n^* \overline{E}) - d\, \mu^{\mathrm{abs}}(\overline{Q}_n)\, (Q_n^{d-1} \cdot \phi_n^* E)}{(Q_n^d)} \qquad \text{for all } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

5.3. **A variant of Theorem 4.8.** In the course of the proof, when applying Corollary 5.3 to produce the lower bound (5.5) we neglected the term

$$\frac{(\widetilde{Q}_n^{d+1})}{\mathrm{vol}(Q_n + \lambda\, \phi_n^* E)}.$$

As it turns out, taking this term into account gives no improvement for an arbitrary adelic divisor $\overline{E}$, though it does when $E = 0$. This leads to the following slight refinement in this situation.

Theorem 5.8. *Assume that there exists a sequence $(\phi_n, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ such that*

$$\lim_{n\to\infty} \frac{1}{r(Q_n; \phi_n^* D)} \Big(\mu^{\mathrm{ess}}(\overline{D}) - \frac{(\overline{Q}_n^{d+1})}{(d+1)(Q_n^d)}\Big) = 0, \quad \sup_{n\in\mathbb{N}} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; \phi_n^* D)} < \infty. \tag{5.6}$$

Then $\overline{D}$ satisfies the equidistribution property at every place $v \in \mathfrak{M}_K$, and its v-adic equidistribution measure is $\nu_{\overline{D},v} = \lim_{n\to\infty} \nu_{n,v}$ with $\nu_{n,v}$ the pushforward to X_v^{an} of the normalized Monge-Ampère measure $c_1(\overline{Q}_{n,v})^{\wedge d}/(Q_n^d)$.

We just outline the proof of this result, as it is almost the same as that of Theorem 4.8. Let $(\phi_n, \overline{Q}_n)_n$ be a sequence satisfying the conditions of Theorem 5.8 and set $\widetilde{Q}_n = \overline{Q}_n - \mu^{\mathrm{abs}}(\overline{Q}_n)\, [\infty]$ for each n. With this notation, the proof of Lemma 5.6 remains valid thanks to the second condition in (5.6), and so

$$\kappa := \sup_{n\in\mathbb{N}} \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{A})}{(Q_n^d)} < \infty.$$

By Proposition 4.6 and Lemma 3.2(2), it suffices to show that

$$-\partial_{-\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \lim_{n\to\infty} \frac{(\overline{Q}_n^d \cdot \phi_n^* \overline{E})}{(Q_n^d)} \tag{5.7}$$

for any $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ over 0. We only treat the case where $\overline{E}$ is DSP, as the general one follows by density as in the proof of Theorem 4.8. Let $\overline{A} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be big and nef with $\overline{A} \pm \overline{E}$ nef. By Corollary 5.3, there exists a constant c_d such that

$$\mu^{\mathrm{ess}}(\overline{D}+\lambda\overline{E}) - \mu^{\mathrm{abs}}(\overline{Q}_n) \geq \frac{(\widetilde{Q}_n^{d+1})}{(d+1)\,(Q_n^d)} + \frac{(\widetilde{Q}_n^d \cdot \phi_n^*\overline{E})}{(Q_n^d)}\,\lambda - \frac{c_d\,\kappa}{r(Q_n;\phi_n^*A)}\,\lambda^2 \tag{5.8}$$

for every n and $0 < \lambda \leq r(Q_n;\phi_n^*A)/2$. Since $E = 0$, by the formula (2.4) we have

$$(\widetilde{Q}_n^{d+1}) = (\overline{Q}_n^{d+1}) - (d+1)\,(Q_n^d)\,\mu^{\mathrm{abs}}(\overline{Q}_n) \quad \text{and} \quad (\widetilde{Q}_n^d \cdot \phi_n^*\overline{E}) = (\overline{Q}_n^d \cdot \phi_n^*\overline{E}).$$

Combining this with (5.8) and dividing by λ gives

$$\frac{\mu^{\mathrm{ess}}(\overline{D}+\lambda\overline{E}) - \mu^{\mathrm{ess}}(\overline{D})}{\lambda} \geq \frac{(\overline{Q}_n^d \cdot \phi_n^*\overline{E})}{(Q_n^d)} + \Big(\frac{(\overline{Q}_n^{d+1})}{(d+1)(Q_n^d)} - \mu^{\mathrm{ess}}(\overline{D})\Big)\frac{1}{\lambda} - \frac{c_d\,\kappa}{r(Q_n;\phi_n^*A)}\,\lambda.$$

As in the proof of Theorem 4.8, a suitable choice of $\lambda = \lambda_n$ permits to conclude that

$$\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) \geq \limsup_{n\to\infty} \frac{(\overline{Q}_n^d \cdot \phi_n^*\overline{E})}{(Q_n^d)}$$

using the first condition in (5.6), and we obtain (5.7) by applying this to $-\overline{E}$.

Remark 5.9. This result gives more flexibility to construct the sequence of semipositive approximations $(\phi_n, \overline{Q}_n)_n$. For example, one can deduce Yuan's equidistribution theorem directly from Theorem 5.8 without using Theorem 2.19.

However, Theorem 5.8 is not more general than Theorem 4.8. In fact, starting with a sequence $(\phi_n, \overline{Q}_n)_n$ satisfying the conditions (5.6) one can modify it to construct another sequence satisfying the condition (4.2), using arguments similar to those in the proof of Lemma 4.13. Since we do not need this in the remainder of the text, we skip the proof of this technical claim.

5.4. **Logarithmic equidistribution.** Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor on X with D big such that there exists a sequence $(\phi_n, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ satisfying the condition of Theorem 4.8. By this result we have that $\overline{D}$ satisfies the equidistribution property at every $v \in \mathfrak{M}_K$ with equidistribution measure

$$\nu_{\overline{D},v} = \lim_{n\to\infty} \nu_{n,v},$$

where $\nu_{n,v}$ denotes the pushforward to X_v^{an} of the normalized v-adic Monge-Ampère measure of $\overline{Q}_n$.

In this section we show that this property extends to functions with logarithmic singularities along effective divisors satisfying a certain numerical condition. Our presentation follows closely that of Chambert-Loir and Thuillier in [CT09], adapting their arguments to our setting.

Definition 5.10. Let E be an effective divisor on X and $v \in \mathfrak{M}_K$. A function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ has *at most logarithmic singularities along* E if it is a real-valued continuous function on $X_v^{\mathrm{an}} \setminus \mathrm{supp}(E)_v^{\mathrm{an}}$ and every $x \in X_v^{\mathrm{an}}$ has a neighborhood $U \subset X_v^{\mathrm{an}}$ together with an equation f_U of E_v^{an} on U and a real number c_U such that

$$|\varphi|_v \leq c_U \log|f_U|_v^{-1} \quad \text{on } U.$$

Equidistribution measures can integrate functions with at most logarithmic singularities along a divisor.

Proposition 5.11. *Let $v \in \mathfrak{M}_K$ and $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ a function with at most logarithmic singularities along an effective divisor on X. Then φ is integrable with respect to $\nu_{\overline{D},v}$.*

For the proof of this proposition we need the next auxiliary result.

Lemma 5.12. *For every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with E effective we have*

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v}.$$

Proof. Let $n \in \mathbb{N}$. By Lemma 2.15 we have

$$h_{\overline{Q}_n}([\phi_n^* E]) \geq d\, \mu^{\mathrm{abs}}(\overline{Q}_n)\, (Q_n^{d-1} \cdot \phi_n^* E), \tag{5.9}$$

where the left-hand side denotes the height with respect to $\overline{Q}_n$ of the $\mathbb{R}$-Weil divisor associated to $\phi_n^* E$. Applying the arithmetic Bézout formula (2.2) we deduce from this

$$\frac{(\overline{Q}_n^d \cdot \phi_n^* \overline{E}) - d\, \mu^{\mathrm{abs}}(\overline{Q}_n)\, (Q_n^{d-1} \cdot \phi_n^* E)}{(Q_n^d)} \geq \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{n,v}.$$

Letting $n \to \infty$, by Theorem 4.8 and Remark 5.7 we have

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \liminf_{n\to\infty} \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{n,v} \geq \sum_{v \in \mathfrak{M}_K} n_v \liminf_{n\to\infty} \int_{X_v^{\mathrm{an}}} \min(c, g_{\overline{E},v})\, d\nu_{n,v}$$

for any $c \in \mathbb{R}$. Since E is effective we have that $g_{\overline{E},v}$ is bounded from below and so $\min(c, g_{\overline{E},v}) \in C(X_v^{\mathrm{an}})$ for every v. Then

$$\sum_{v \in \mathfrak{M}_K} n_v \liminf_{n\to\infty} \int_{X_v^{\mathrm{an}}} \min(c, g_{\overline{E},v})\, d\nu_{n,v} = \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} \min(c, g_{\overline{E},v})\, d\nu_{\overline{D},v}$$

by the equidistribution property at every place. The statement follows by letting $c \to \infty$ and applying the monotone convergence theorem. □

Proof of Proposition 5.11. Since X_v^{an} is compact, we can assume without loss of genericity that $\varphi = g_{\overline{E},v}$ for an adelic divisor $\overline{E}$ over an effective $E \in \mathrm{Div}(X)$. Up to adding an adelic divisor over $0 \in \mathrm{Div}(X)$ we can furthermore assume that $\overline{E}$ is effective. In this situation we have $g_{\overline{E},w} \geq 0$ for every $w \in \mathfrak{M}_K$ and so Lemma 5.12 implies

$$\infty > \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v},$$

and so $g_{\overline{E},v}$ is integrable with respect to $\nu_{\overline{D},v}$. □

The following is the main result of this section.

Theorem 5.13. *Let $\overline{E} \in \widehat{\mathrm{Div}}(X)$ with E effective such that*

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v}. \tag{5.10}$$

Then for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ and $v \in \mathfrak{M}_K$ we have

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \varphi\, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \varphi\, d\nu_{\overline{D},v}$$

for any function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ with at most logarithmic singularities along E.

Proof. By Proposition 5.11 and [CT09, Lemma 6.3] it suffices to consider the case $\varphi = g_{\overline{E},v}$. Then by Proposition 4.3 and Theorem 4.8 we have

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \lim_{\ell\to\infty} \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v},$$

and so the condition (5.10) implies

$$\begin{aligned}\sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v} &= \lim_{\ell\to\infty} \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v} \\ &\geq \sum_{v\in\mathfrak{M}_K} n_v \liminf_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v}. \qquad (5.11)\end{aligned}$$

On the other hand, for every v and any $c \in \mathbb{R}$ we have $\min(c, g_{\overline{E},v}) \in C(X_v^{\mathrm{an}})$ and so

$$\liminf_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v} \geq \liminf_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \min(c, g_{\overline{E},v})\, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \min(c, g_{\overline{E},v})\, d\nu_{\overline{D},v}$$

by the equidistribution property. Letting $c \to \infty$ we get that

$$\liminf_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v} \geq \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v}.$$

Summing up these inequalities over all the places and taking (5.11) into account we deduce that they are in in fact equalities. We conclude by observing that such equality remains true when $(x_\ell)_\ell$ is replaced by an arbitrary ¡subsequence, since the latter remains generic and $\overline{D}$-small. Therefore

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v}$$

as desired. □

Remark 5.14. The condition (5.10) is independent of the choice of the adelic structure over E. Indeed, let $\overline{E}'$ be another adelic divisor over E. Then there exists a finite set $\mathfrak{S} \subset \mathfrak{M}_K$ and a collection $\varphi_v \in C(X_v^{\mathrm{an}})^{G_v}$, $v \in \mathfrak{S}$, such that $\overline{E}' - \overline{E} = \sum_{v\in\mathfrak{S}} \overline{0}^{\varphi_v}$. Since the essential minimum function is differentiable at $\overline{D}$, we have

$$\partial_{\overline{E}'}\, \mu^{\mathrm{ess}}(\overline{D}) = \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) + \sum_{v\in\mathfrak{S}} \partial_{\overline{0}^{\varphi_v}}\, \mu^{\mathrm{ess}}(\overline{D}).$$

Setting $\varphi_v = 0$ for $v \in \mathfrak{M}_K \setminus \mathfrak{S}$, Proposition 4.6 together with (5.10) then gives

$$\partial_{\overline{E}'}\, \mu^{\mathrm{ess}}(\overline{D}) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} (g_{\overline{E},v} + \varphi_v)\, d\nu_{\overline{D},v} = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E}',v}\, d\nu_{\overline{D},v}$$

and so $\overline{E}'$ also verifies this condition.

If the sequences of probability measures approaching the equidistribution measures are eventually constant, we can rephrase the condition in Theorem 5.13 in terms of the gaps in Zhang's lower bound for the heights of Weil divisors in (5.9).

Corollary 5.15. *Assume that for every $v \in \mathfrak{M}_K$ the sequence of probability measures $(\nu_{n,v})_n$ is eventually constant, and let E be an effective divisor on X such that*

$$\lim_{n\to\infty} \frac{h_{\overline{Q}_n}([\phi_n^* E]) - d\,\mu^{\mathrm{abs}}(\overline{Q}_n)\,(Q_n^{d-1}\cdot \phi_n^* E)}{(Q_n^d)} = 0.$$

Then for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ and $v \in \mathfrak{M}_K$ we have

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \varphi\, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \varphi\, d\nu_{\overline{D},v}$$

for any function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R}\cup\{\pm\infty\}$ with at most logarithmic singularities along E.

Proof. Let $\overline{E}$ be an adelic divisor over E. By Remark 5.7 and the arithmetic Bézout formula (2.2) we have

$$\begin{aligned}\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) &= \lim_{n\to\infty} \Big(\frac{h_{\overline{Q}_n}([\phi_n^* E]) - d\,\mu^{\mathrm{abs}}(\overline{Q}_n)\,(Q_n^{d-1}\cdot \phi_n^* E)}{(Q_n^d)} + \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{n,v}\Big)\\ &= \lim_{n\to\infty} \frac{h_{\overline{Q}_n}([\phi_n^* E]) - d\,\mu^{\mathrm{abs}}(\overline{Q}_n)\,(Q_n^{d-1}\cdot \phi_n^* E)}{(Q_n^d)} + \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v},\end{aligned}$$

and so $\overline{E}$ verifies the condition (5.10). □

Remark 5.16. The assumption in Corollary 5.15 that the sequences of probability measures $(\nu_{n,v})_n$ are eventually constant is verified in the setting of dynamical systems (Theorem 7.9). It would be interesting to know whether this corollary remains valid without this technical assumption.

Corollary 5.15 allows to recover the logarithmic equidistribution theorem from [CT09, Theorem 1.2] as follows. Let $\overline{D}$ be a semipositive adelic $\mathbb{R}$-divisor on X with D ample such that

$$\mu^{\mathrm{ess}}(\overline{D}) = \frac{(\overline{D}^{d+1})}{(d+1)(D^d)}, \tag{5.12}$$

and let $\overline{E} \in \widehat{\mathrm{Div}}(X)$ with E is effective such that

$$\frac{h_{\overline{D}}([E])}{d\,(D^{d-1}\cdot E)} = \mu^{\mathrm{ess}}(\overline{D}).$$

By Theorem 2.19 the equality (5.12) implies $\mu^{\mathrm{ess}}(\overline{D}) = \mu^{\mathrm{abs}}(\overline{D})$. Therefore the condition of Corollary 5.15 is trivially satisfied for the constant sequence $(\phi_n, \overline{Q}_n) = (\mathrm{Id}_X, \overline{D})$, $n \in \mathbb{N}$, thus giving the stated equidistribution for functions with at most logarithmic singularities along E.

5.5. **A partial converse.** The next result answers Question 4.19 under an additional technical assumption, roughly saying that $\overline{D}$ has a suitable upper bound for which Zhang's inequality is an equality. As we will see in Section 6, this assumption is always satisfied for semipositive toric adelic $\mathbb{R}$-divisors (Proposition 6.10), which will allow us to give an affirmative answer to this question in this setting (Theorem 6.9). As before, we denote by $\overline{D}$ an adelic $\mathbb{R}$-divisor on X with D big.

Proposition 5.17. *Assume that there exists a semipositive adelic $\mathbb{R}$-divisor $\overline{D}'$ over D such that $\overline{D}' - \overline{D}$ is pseudo-effective and $\mu^{\mathrm{ess}}(\overline{D}') = \mu^{\mathrm{abs}}(\overline{D}') = \mu^{\mathrm{ess}}(\overline{D})$. Then the following conditions are equivalent:*

(1) $\displaystyle\lim_{t\to\mu^{\mathrm{ess}}(\overline{D})}\frac{\mu^{\mathrm{ess}}(\overline{D})-t}{\rho(\overline{D}(t))}=0$,

(2) the essential minimum function is differentiable at $\overline{D}$*,*

(3) $\overline{D}$ *has the equidistribution property at every place* $v\in\mathfrak{M}_K$*.*

We deduce this result as a special case of the next lemma.

Lemma 5.18. *Assume that there exists a nef* $\overline{A}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ *with* A *big such that* $\partial_{\overline{A}}\,\mu^{\mathrm{ess}}(\overline{D})=0$*. Then the following conditions are equivalent:*

(1) $\displaystyle\lim_{t\to\mu^{\mathrm{ess}}(\overline{D})}\frac{\mu^{\mathrm{ess}}(\overline{D})-t}{\rho(\overline{D}(t))}=0$,

(2) the essential minimum function is differentiable at $\overline{D}$*,*

(3) $\partial_{-\overline{A}}\,\mu^{\mathrm{ess}}(\overline{D})=0$.

If moreover $A=D$*, then they are equivalent to the condition:*

(4) $\overline{D}$ *has the equidistribution property at every place* $v\in\mathfrak{M}_K$*.*

Proof. We have (1) ⇒ (2) by Theorem 4.16 and (2) ⇒ (3) is trivial. We next show that (3) ⇒ (1). If (3) holds then by Lemma 4.18

$$\lim_{t\to\mu^{\mathrm{ess}}(\overline{D})}\frac{(\langle\overline{D}(t)^d\rangle\cdot\overline{A})}{\mathrm{vol}(R^t(\overline{D}))}=0. \tag{5.13}$$

Fix a real number $t<\mu^{\mathrm{ess}}(\overline{D})$ and let $(\phi_n,\overline{P}_n)_n$ be a Fujita approximation sequence of $\overline{D}(t)$. Then by Proposition 3.6

$$\lim_{n\to\infty}\frac{(\overline{P}_n^d\cdot\overline{A})}{(P_n^d)}=\frac{(\langle\overline{D}(t)^d\rangle\cdot\overline{A})}{\mathrm{vol}(R^t(\overline{D}))}\quad\text{and}\quad\lim_{n\to\infty}\mu^{\mathrm{ess}}(\overline{P}_n)=\mu^{\mathrm{ess}}(\overline{D})-t. \tag{5.14}$$

Set $\overline{P}'_n=\overline{P}_n-\mu^{\mathrm{ess}}(\overline{P}_n)\,[\infty]$, $n\in\mathbb{N}$. Then $\overline{P}'_n$ is pseudo-effective by Theorem 2.17, and by the formula (2.4) we have

$$\frac{(\overline{P}_n^d\cdot\overline{A})}{(P_n^d)}=\frac{(\overline{P}_n^{d-1}\cdot\overline{A}\cdot\overline{P}'_n)}{(P_n^d)}+\mu^{\mathrm{ess}}(\overline{P}_n)\,\frac{(P_n^{d-1}\cdot A)}{(P_n^d)}.$$

Since $\overline{P}_n$ and $\overline{A}$ are nef, by Lemma 2.13 the first summand is non-negative and so

$$\frac{(\overline{P}_n^d\cdot\overline{A})}{(P_n^d)}\geq\mu^{\mathrm{ess}}(\overline{P}_n)\,\frac{(P_n^{d-1}\cdot A)}{(P_n^d)}\geq\frac{\mu^{\mathrm{ess}}(\overline{P}_n)}{d\,r(P_n;A)}\geq\frac{\mu^{\mathrm{ess}}(\overline{P}_n)}{d\,\rho(\overline{D}(t);A)},$$

where the second inequality follows from Lemma 1.3 and the third from the definition of the inradius of $\overline{D}(t)$ with respect to A. Letting $n\to\infty$ and applying (5.14) we get

$$\frac{(\langle\overline{D}(t)^d\rangle\cdot\overline{A})}{\mathrm{vol}(R^t(\overline{D}))}\geq\frac{\mu^{\mathrm{ess}}(\overline{D})-t}{d\,\rho(\overline{D}(t);A)}.$$

By Lemma 1.2 there is $c>0$ such that $\rho(\overline{D}(t);A)\leq c\,\rho(\overline{D}(t);D)=c\,\rho(\overline{D}(t))$ for every $t<\mu^{\mathrm{ess}}(\overline{D})$. Therefore (1) follows by letting $t\to\mu^{\mathrm{ess}}(\overline{D})$ and using (5.13).

For the last claim, by Proposition 4.6 and Lemma 3.2(2) the condition (4) is equivalent to the fact that the essential minimum function is differentiable along the subspace of adelic divisors on X over the zero divisor. In particular, it is implied by (2).

Now assume that $A=D$ and that (4) holds. Then $\overline{E}\coloneqq\overline{A}-\overline{D}$ is an adelic divisor over $E=0$ and so the essential minimum function is differentiable at $\overline{D}$ along $\overline{E}$. Since this function is clearly differentiable along $\overline{D}$, by Lemma 3.2(2) it is also differentiable along $\overline{A}$. This gives (3). □

Proof of Proposition 5.17. Let $\overline{A} = \overline{D}' - \mu^{\mathrm{ess}}(\overline{D})\,[\infty]$. Then $\mu^{\mathrm{ess}}(\overline{A}) = \mu^{\mathrm{abs}}(\overline{A}) = 0$, and in particular $\overline{A}$ is nef. Thus by Lemma 5.18 it suffices to show that $\partial_{\overline{A}}\,\mu^{\mathrm{ess}}(\overline{D}) = 0$. This is clear, since by Lemma 2.16(1) for every $\lambda > 0$ we have

$$\mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{A}) \geq \mu^{\mathrm{ess}}(\overline{D}) + \lambda\mu^{\mathrm{ess}}(\overline{A}) = \mu^{\mathrm{ess}}(\overline{D}),$$

whereas Lemma 2.16(4) gives the converse inequality, namely

$$\mu^{\mathrm{ess}}(\overline{D}) = \mu^{\mathrm{ess}}(\overline{D}') = \mu^{\mathrm{ess}}(\overline{D}' + \lambda\overline{D}') - \lambda\mu^{\mathrm{ess}}(\overline{D}') = \mu^{\mathrm{ess}}(\overline{D}' + \lambda\overline{A}) \geq \mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{A}).$$

□

6. Toric varieties

Here we study the differentiability of the essential minimum function in the toric setting and its consequences for the equidistribution of the Galois orbits of small generic sequences of algebraic points. To this end, first we review the algebraic and Arakelov geometries of toric varieties following [BPS14, BPS15, BMPS16, BPRS19] and study the inradii and positive intersection numbers of (adelic) $\mathbb{R}$-divisors on toric varieties. We then prove our toric differentiability results (Theorems 6.6 and 6.9) and extend the corresponding equidistribution properties to test functions with logarithmic singularities along special hypersurfaces (Theorem 6.12 and Corollary 6.13).

6.1. **Geometric and arithmetic aspects.** Let $\mathbb{T} \simeq \mathbb{G}_{\mathrm{m}}^d$ be a split d-dimensional torus over K and set

$$M = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_{\mathrm{m}}) \quad \text{and} \quad N = \mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, \mathbb{T})$$

for its lattices of characters and of co-characters. These are both isomorphic to $\mathbb{Z}^d$ and dual of each other, that is $M = N^\vee$ and $N = M^\vee$. Set then $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. These vector spaces are also dual of each other, and for $u \in N_\mathbb{R}$ and $x \in M_\mathbb{R}$ we denote their pairing by $\langle u, x\rangle$. Let also $K[M]$ be the group algebra of M, and for each $m \in M$ let $\chi^m \in K[M]$ be the corresponding monomial.

Let X be a *projective toric variety* with torus $\mathbb{T}$ and D a *toric* $\mathbb{R}$-*divisor* on it. By this we mean a normal projective variety over K containing $\mathbb{T}$ as an open subset and equipped with an action of this torus extending its action onto itself by translations, together with an $\mathbb{R}$-divisor that is invariant under this action.

Classically toric varieties and $\mathbb{R}$-divisors are constructed and classified with polyhedral objects. Thus to X and D respectively correspond a *fan* Σ_X and an $\mathbb{R}$-*virtual support function* Ψ_D. The fan Σ_X is a polyhedral complex of strongly convex cones defined over N covering the whole of $N_\mathbb{R}$, whereas the $\mathbb{R}$-virtual support function Ψ_D is a real-valued function on $N_\mathbb{R}$ that is linear on each of the cones of this fan. We also associate to D the subset of $M_\mathbb{R}$ defined as

$$\Delta_D = \{x \in M_\mathbb{R} \mid \langle u, x\rangle \geq \Psi_D(u) \text{ for every } u \in N_\mathbb{R}\}.$$

It is a *quasi-rational polytope*, that is a polytope with rational slopes.

The positivity invariants and properties of D can be read from its $\mathbb{R}$-virtual support function and polytope. For instance, the volume of D is given by

$$\mathrm{vol}(D) = d!\,\mathrm{vol}_M(\Delta_D)$$

where vol_M denotes the Haar measure on $M_{\mathbb{R}}$ normalized so that M has covolume 1. In particular, if D is nef then $(D^d) = d!\,\mathrm{vol}_M(\Delta_D)$. More generally, for a family D_i, $i = 1, \dots, d$, of nef toric $\mathbb{R}$-divisors on X we have

$$(D_1 \cdots D_d) = \mathrm{MV}_M(\Delta_{D_1}, \dots, \Delta_{D_d}), \tag{6.1}$$

where MV_M denotes the mixed volume function with respect to vol_M.

The $\mathbb{R}$-divisor D is pseudo-effective if and only if $\Delta_D \neq \emptyset$, and is big if and only if $\dim(\Delta_D) = d$. In addition, D is nef if and only if Ψ_D is concave. For a nef toric $\mathbb{R}$-divisor E we have that $D - E$ is pseudo-effective if and only if there exists $x \in M_{\mathbb{R}}$ such that $x + \Delta_E \subset \Delta_D$. All of this can be found in [BMPS16, Section 4].

To study the arithmetic counterpart of these constructions and results, for each place $v \in \mathfrak{M}_K$ we denote by $\mathbb{S}_v$ the *compact torus* of the v-adic analytic torus $\mathbb{T}_v^{\mathrm{an}}$ [BPS14, Section 4.2]. In the Archimedean case $\mathbb{S}_v$ is isomorphic to the real torus $(S^1)^d$, whereas in the non-Archimedean case it is an analytic subgroup of $\mathbb{T}_v^{\mathrm{an}}$ in the sense of Berkovich. We also consider the *valuation map*

$$\mathrm{val}_v \colon \mathbb{T}_v^{\mathrm{an}} \longrightarrow N_{\mathbb{R}}.$$

With a splitting of the torus, we can identify the dense subset $\mathbb{T}_v^{\mathrm{an}}(\mathbb{C}_v)$ with $(\mathbb{C}_v^\times)^d$ and the vector space $N_{\mathbb{R}}$ with $\mathbb{R}^d$. In these coordinates, the valuation map writes down as

$$\mathrm{val}_v(x_1, \dots, x_d) = (-\log|x_1|_v, \dots, -\log|x_d|_v).$$

Now let $\overline{D}$ be a *toric adelic $\mathbb{R}$-divisor* on X, that is an adelic $\mathbb{R}$-divisor on X whose geometric $\mathbb{R}$-divisor D is toric and whose v-adic Green function $g_{\overline{D},v}$ is invariant under the action of $\mathbb{S}_v$ for every v. Toric adelic $\mathbb{R}$-divisors over D can be constructed and classified with adelic families of functions on $N_{\mathbb{R}}$ whose behavior at infinity is governed by the $\mathbb{R}$-virtual support function Ψ_D [BPS14, Proposition 4.3.10], [BMPS16, Proposition 4.16]. Accordingly we denote by

$$\psi_{\overline{D},v} \colon N_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad v \in \mathfrak{M}_K,$$

the family of *metric functions* associated to $\overline{D}$. For each v, the v-adic metric function is defined as

$$\psi_{\overline{D},v}(u) = -g_{\overline{D},v}(x) \quad \text{for every } u \in N_{\mathbb{R}} \text{ and } x \in \mathrm{val}_v^{-1}(u). \tag{6.2}$$

It is continuous and has bounded difference with respect to Ψ_D for every v, and it is equal to Ψ_D for all but a finite number of places.

We also associate to $\overline{D}$ its family of *local roof functions*

$$\vartheta_{\overline{D},v} \colon \Delta_D \longrightarrow \mathbb{R}, \quad v \in \mathfrak{M}_K.$$

For each v, the v-adic roof function is a continuous concave function on the polytope that is defined as

$$\vartheta_{\overline{D},v}(x) = \inf_{u \in N_{\mathbb{R}}} \langle u, x \rangle - \psi_{\overline{D},v}(u) \quad \text{for every } x \in \Delta_D.$$

These functions are zero for all but a finite number of places. We consider then the *global roof function* $\vartheta_{\overline{D}} \colon \Delta_D \to \mathbb{R}$, defined as the weighted sum

$$\vartheta_{\overline{D}} = \sum_{v \in \mathfrak{M}_K} n_v \vartheta_{\overline{D},v}.$$

We also consider the compact convex set where this concave function is nonnegative:

$$\Gamma_{\overline{D}} = \{x \in \Delta_D \mid \vartheta_{\overline{D}}(x) \geq 0\}. \tag{6.3}$$

In analogy with the geometric case, the positivity invariants and properties of $\overline{D}$ can be read from its metric and roof functions. For instance, the essential minimum of $\overline{D}$ is the maximum of its global roof function [BPS15, Theorem 1.1]:

$$\mu^{\mathrm{ess}}(\overline{D}) = \max_{x\in\Delta_D} \vartheta_{\overline{D}}(x). \tag{6.4}$$

Moreover, if $\overline{D}$ is semipositive then $\mu^{\mathrm{abs}}(\overline{D}) = \min_{x\in\Delta_D} \vartheta_{\overline{D}}(x)$ [BPS15, Remark 3.15].

The volumes of $\overline{D}$ can be computed as [BMPS16, Theorem 5.6]

$$\widehat{\mathrm{vol}}(\overline{D}) = (d+1)! \int_{\Gamma_{\overline{D}}} \vartheta_{\overline{D}}\, d\mathrm{vol}_M \quad \text{and} \quad \widehat{\mathrm{vol}}_{\chi}(\overline{D}) = (d+1)! \int_{\Delta_D} \vartheta_{\overline{D}}\, d\mathrm{vol}_M\,.$$

In particular, if $\overline{D}$ is semipositive then $(\overline{D}^{d+1}) = (d+1)! \int_{\Delta_D} \vartheta_{\overline{D}}\, d\mathrm{vol}_M$. More generally, the arithmetic intersection number of a family of semipositive toric adelic $\mathbb{R}$-divisors $\overline{D}_i$, $i = 0, \dots, d$, can be computed as

$$(\overline{D}_0 \cdots \overline{D}_d) = \sum_{v\in\mathfrak{M}_K} n_v \,\mathrm{MI}_M(\vartheta_{\overline{D}_0,v}, \dots, \vartheta_{\overline{D}_d,v}), \tag{6.5}$$

where MI_M denotes the mixed integral function with respect to the Haar measure vol_M on $M_{\mathbb{R}}$ [BPS14, Theorem 5.2.5].

We have that $\overline{D}$ is semipositive if and only if $\psi_{\overline{D},v}$ is concave for every v [BMPS16, Proposition 4.19]. By Theorem 6.1 in *loc. cit.* we have that $\overline{D}$ is pseudo-effective if and only if there exists $x \in \Delta_D$ such that $\vartheta_{\overline{D}}(x) \geq 0$ or equivalently, if and only if $\Gamma_{\overline{D}} \neq \emptyset$. We also have that $\overline{D}$ is big if and only if $\dim(\Delta_D) = d$ and there exists $x \in \Delta_D$ such that $\vartheta_{\overline{D}}(x) > 0$, in which case $\Gamma_{\overline{D}}$ is a convex body. By the same result, when $\overline{D}$ is semipositive then it is nef if and only of $\vartheta_{\overline{D}}(x) \geq 0$ for every $x \in \Delta_D$.

For a semipositive adelic $\mathbb{R}$-divisor $\overline{E}$ on X we have that $\overline{D} - \overline{E}$ is pseudo-effective if and only if $\Delta_E \subset \Delta_D$ and $\vartheta_{\overline{E},v}(x) \leq \vartheta_{\overline{D},v}(x)$ for every $v \in \mathfrak{M}_K$ and $x \in \Delta_E$ [BMPS16, Proposition 6.4 and Theorem 7.2(1)].

If $\overline{D}$ is big, then for any sequence $(\Lambda_n)_n$ of quasi-rational polytopes uniformly approaching the convex body $\Gamma_{\overline{D}}$ from inside one can construct a Fujita approximation sequence of $\overline{D}$

$$(\phi_n \colon X_n \to X, \overline{P}_n)_n \tag{6.6}$$

such that both ϕ_n and $\overline{P}_n$ are toric for each n. The modification $\phi_n \colon X_n \to X$ is *toric* if X_n is also a toric variety with the same torus $\mathbb{T}$ and the restriction of ϕ_n to this torus is the identity. At the combinatorial level, a toric modification corresponds to a (regular) refinement of the fan Σ_X. On the other hand, $\overline{P}_n$ is a toric adelic $\mathbb{R}$-divisor on X_n with polytope equal to Λ_n and local roof functions equal to those of $\overline{D}$ restricted to this polytope, that is

$$\Delta_{P_n} = \Lambda_n \quad \text{and} \quad \vartheta_{\overline{P}_n,v} = \vartheta_{\overline{D},v}\big|_{\Lambda_n} \quad \text{for all } v \in \mathfrak{M}_K. \tag{6.7}$$

This is explained in [BMPS16, Theorem 7.2] and its proof.

6.2. **Inradii and positive intersection numbers.** In [Tei82], Teissier first pointed out the relation between the inradius of toric line bundles and the inradius of the associated polytopes in the sense of convex geometry. The next statement puts this observation into the setting of $\mathbb{R}$-divisors.

Proposition 6.1. *Let D and A be toric $\mathbb{R}$-divisors on X such that D is big and A is big and nef. Then*
$$r(D;A) = r(\Delta_D;\Delta_A),$$
where $r(\Delta_D;\Delta_A)$ denotes the inradius in the sense of Definition A.1.

Proof. Since A is nef, for each $\lambda \in \mathbb{R}$ we have that $D - \lambda A$ is pseudo-effective if and only if there exists $x \in M_{\mathbb{R}}$ such that $x + \lambda\,\Delta_A \subset \Delta_D$. The equality between the two inradii follows then directly from their definitions. $\square$

Now let $\overline{D}$ be a big toric adelic $\mathbb{R}$-divisor on the projective toric variety X. From the existence and properties of its toric Fujita approximation sequences (6.6) we will derive both a lower bound for the inradius of $\overline{D}$ and a formula for its arithmetic positive intersection numbers.

Proposition 6.2. *Let A be a toric $\mathbb{R}$-divisor on X that is big and nef. Then*
$$\rho(\overline{D};A) \geq r(\Gamma_{\overline{D}};\Delta_A).$$

Proof. Let $(\phi_n, \overline{P}_n)$ be a toric Fujita approximation sequence of $\overline{D}$ as in (6.6). With notation as in Definition 4.15 we have $(\phi_n, \overline{P}_n) \in \Theta(\overline{D})$ for every n. Then
$$\rho(\overline{D};A) \geq \sup_{n\in\mathbb{N}} r(P_n;\phi_n^*A) = \sup_{n\in\mathbb{N}} r(\Lambda_n;\Delta_A) = r(\Gamma_{\overline{D}};\Delta_A)$$
by Proposition 6.1 and the fact that the sequence of polytopes $(\Lambda_n)_n$ approaches $\Gamma_{\overline{D}}$ from inside. $\square$

Proposition 6.3. *Let $\overline{E}$ be a semipositive toric adelic $\mathbb{R}$-divisor on X. Then*
$$\mathrm{vol}(R^0(\overline{D})) = d!\,\mathrm{vol}(\Gamma_{\overline{D}}), \quad (\langle \overline{D}^d\rangle \cdot \overline{E}) = \sum_{v\in\mathfrak{M}_K} n_v\,\mathrm{MI}_M(\vartheta_{\overline{D},v}|_{\Gamma_{\overline{D}}}, \dots, \vartheta_{\overline{D},v}|_{\Gamma_{\overline{D}}}, \vartheta_{\overline{E},v}).$$

Proof. Let $(\phi_n, \overline{P}_n)$ be a toric Fujita approximation sequence of $\overline{D}$ as in (6.6). By Proposition 3.6 and the formula for toric intersection numbers (6.1) we have
$$\mathrm{vol}(R^0(\overline{D})) = \lim_{n\to\infty} (P_n^d) = \lim_{n\to\infty} d!\,\mathrm{vol}_M(\Lambda_n) = d!\,\mathrm{vol}(\Gamma_{\overline{D}}).$$

For the second formula, by Definition 3.7 we have $(\langle \overline{D}^d\rangle \cdot \overline{E}) = \lim_{n\to\infty}(\overline{P}_n^d \cdot \phi_n^*\overline{E})$, whereas by (6.7) and the formula for toric arithmetic intersection numbers (6.5) we get
$$(\overline{P}_n^d \cdot \phi_n^*\overline{E}) = \sum_{v\in\mathfrak{M}_K} n_v\,\mathrm{MI}_M(\vartheta_{\overline{D},v}|_{\Delta_n}, \dots, \vartheta_{\overline{D},v|_{\Delta_n}}, \vartheta_{\overline{E},v}) \quad \text{for every } n \in \mathbb{N}.$$
We conclude by taking the limit $n \to \infty$ and applying Lemma A.9. $\square$

Following [BPS14, Definition 4.3.3], given an arbitrary adelic structure over a toric $\mathbb{R}$-divisor one can construct a toric one by an averaging process. To describe it, for each $v \in \mathfrak{M}_K$ and $u \in N_{\mathbb{R}}$ we recall the probability measure $\eta_{v,u}$ on X_v^{an} from [BPRS19, Definition 5.1], which is defined as:

(1) if $v \in \mathfrak{M}_K^\infty$ then $\eta_{v,u}$ is the translation by any point $x \in \mathrm{val}_v^{-1}(u) \subset \mathbb{T}_v^{\mathrm{an}} \simeq (\mathbb{C}^\times)^d$ of the Haar probability measure of the compact torus $\mathbb{S}_v \simeq (S^1)^d$,

(2) if $v \in \mathfrak{M}_K \backslash \mathfrak{M}_K^\infty$ then $\eta_{v,u}$ is the Dirac measure at the point $\zeta_v(u) \in \mathrm{val}_v^{-1}(u) \subset \mathbb{T}_v^{\mathrm{an}}$ corresponding to the multiplicative seminorm on $K[M]$ defined as
$$|f|_{\zeta_v(u)} = \max_{m\in M} |\alpha_m|_v\, e^{-\langle u,m\rangle} \quad \text{for every } f = \sum_{m\in M} \alpha_m \chi^m \in K[M].$$

Definition 6.4. Let $\overline{E}$ be an adelic $\mathbb{R}$-divisor on X with E toric. For each $v \in \mathfrak{M}_K$ let $\widehat{g}_v \colon X_v^{\mathrm{an}} \setminus E_v^{\mathrm{an}} \to \mathbb{R}$ be the function defined as

$$\widehat{g}_v(x) = \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\eta_{v,\mathrm{val}_v(x)}$$

and set $\overline{E}^{\mathrm{tor}} = (E, (\widehat{g}_v)_{v\in\mathfrak{M}_K})$. This is a toric adelic $\mathbb{R}$-divisor on X.

We need the following invariance of arithmetic positive intersection numbers with respect to this averaging process.

Proposition 6.5. *For every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ we have $(\langle \overline{D}^d\rangle \cdot \overline{E}) = (\langle \overline{D}^d\rangle \cdot \overline{E}^{\mathrm{tor}})$.*

Proof. Let $\overline{P}$ be a toric adelic $\mathbb{R}$-divisor on X. With notations as in Definition 6.4, by the arithmetic Bézout formula we have

$$(\overline{P}^d \cdot \overline{E}) - (\overline{P}^d \cdot \overline{E}^{\mathrm{tor}}) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} (g_{\overline{E},v} - \widehat{g}_v)\, c_1(\overline{P}_v)^{\wedge d}. \tag{6.8}$$

Let $v \in \mathfrak{M}_K$. If v is Archimedean then

$$\begin{aligned}\int_{X_v^{\mathrm{an}}} \widehat{g}_v\, c_1(\overline{P}_v)^{\wedge d} &= \int_{X_v^{\mathrm{an}}} \left(\int_{\mathbb{S}_v} g_{\overline{E},v}(t\cdot x)\, d\eta_{v,0}(t)\right) c_1(\overline{P}_v)^{\wedge d}(x)\\ &= \int_{\mathbb{S}_v} \left(\int_{X_v^{\mathrm{an}}} g_{\overline{E},v}(t\cdot x)\, c_1(\overline{P}_v)^{\wedge d}(x)\right) d\eta_{v,0}(t) = \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, c_1(\overline{P}_v)^{\wedge d}\end{aligned}$$

by Fubini's theorem and the invariance of the v-adic Monge-Ampère measure of $\overline{P}$ under the action of $\mathbb{S}_v$. On the other hand, when v is non-Archimedean we have

$$\int_{X_v^{\mathrm{an}}} \widehat{g}_v\, c_1(\overline{P}_v)^{\wedge d} = \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}(\zeta_v(\mathrm{val}_v(x)))\, c_1(\overline{P}_v)^{\wedge d} = \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}(x)\, c_1(\overline{P}_v)^{\wedge d}$$

by the characterization of the Monge-Ampère measures of semipositive toric adelic divisors in [BPS14, Theorem 4.8.11]. Combining this with (6.8) we get

$$(\overline{P}^d \cdot \overline{E}) = (\overline{P}^d \cdot \overline{E}^{\mathrm{tor}}). \tag{6.9}$$

Now let $(\phi_n, \overline{P}_n)_n$ be a Fujita approximation sequence of $\overline{D}$ as in (6.6). By (6.9)

$$(\overline{P}_n^d \cdot \phi_n^*\overline{E}) = (\overline{P}_n^d \cdot (\phi_n^*\overline{E})^{\mathrm{tor}}) = (\overline{P}_n^d \cdot \phi_n^*(\overline{E}^{\mathrm{tor}})) \quad \text{for every } n \in \mathbb{N}$$

because $\overline{P}_n$ is toric and the averaging process commutes with the toric modification ϕ_n. Indeed, this process occurs on the open subset $\mathbb{T} \subset X, X_n$, which remains unchanged under this modification. We conclude by taking the limit as $n \to \infty$. □

6.3. **Equidistribution on toric varieties.** Let X be a projective toric variety with torus $\mathbb{T}$ and $\overline{D}$ a toric adelic $\mathbb{R}$-divisor on X with D big. For each $t \leq \mu^{\mathrm{ess}}(\overline{D})$ we set

$$S_t(\vartheta_{\overline{D}}) = \{x \in \Delta_D \mid \vartheta_{\overline{D}}(x) \geq t\}$$

for the corresponding sup-level set of the global roof function of $\overline{D}$. It is a nonempty compact convex subset of Δ_D that is d-dimensional whenever $t < \mu^{\mathrm{ess}}(\overline{D})$. Set also $\Delta_{D,\max} = S_{\mu^{\mathrm{ess}}(\overline{D})}(\vartheta_{\overline{D}})$.

The global roof function is said to be *wide* if after fixing an arbitrary norm on $M_{\mathbb{R}}$, the width of these sup-level sets remains relatively large as the level approaches its maximum (Definition A.6). By Proposition A.3, this is equivalent to the fact that the inradius of these sup-level sets with respect to any fixed convex body remains relatively large as the level approaches its maximum. By the same result, it is also equivalent to the fact that for any $x_0 \in \Delta_{D,\max}$ we have that $0 \in N_{\mathbb{R}}$ is a vertex of the sup-differential $\partial\vartheta_{\overline{D}}(x_0) \subset N_{\mathbb{R}}$ (Definition A.2). When this condition holds, by Proposition A.8 one can associate to $\vartheta_{\overline{D}}$ a unique *balanced family of sup-gradients* with respect to its decomposition into local roof functions. This is a family of vectors

$$u_v \in N_{\mathbb{R}}, \quad v \in \mathfrak{M}_K,$$

such that $u_v \in \partial\vartheta_{\overline{D},v}(x_0)$ for every v with $u_v = 0$ for all but a finite number of places and verifying the balancing condition $\sum_{v\in\mathfrak{M}_K} n_v u_v = 0$. We let

$$\eta_{\overline{D},v} = \eta_{v,u_v}$$

be the probability measure on X_v^{an} from Section 6.2 for the point $u_v \in N_{\mathbb{R}}$.

The following is the main result of this section. It is an application of Theorem 4.8, or rather of its reformulation in Theorem 4.16 in terms of arithmetic positive intersection numbers, together with the constructions and results from Sections 6.1 and 6.2 and from Appendix A.

Theorem 6.6. *If $\vartheta_{\overline{D}}$ is wide then the essential minimum function is differentiable at $\overline{D}$ and*

$$\partial_{\overline{E}}\mu^{\mathrm{ess}}(\overline{D}) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\eta_{\overline{D},v} \qquad \textit{for every } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}} \textit{ with } E \textit{ toric.}$$

In particular, in this case $\overline{D}$ satisfies the equidistribution property at every $v \in \mathfrak{M}_K$ with $\nu_{\overline{D},v} = \eta_{\overline{D},v}$.

Proof. Let $(u_v)_v$ be the balanced family of sup-gradients of $\vartheta_{\overline{D}}$, so that $\eta_{\overline{D},v} = \eta_{v,u_v}$ for every v. Set $\mu = \mu^{\mathrm{ess}}(\overline{D})$ for short. Then

$$\lim_{t\to\mu} \frac{\mu - t}{r(S_t(\vartheta_{\overline{D}}); \Delta_D)} = 0. \tag{6.10}$$

For each $t < \mu$ the global roof functions of $\overline{D}$ and its shift by t are related by $\vartheta_{\overline{D}} = \vartheta_{\overline{D}(t)} + t$, and so $S_t(\vartheta_{\overline{D}}) = \Gamma_{\overline{D}(t)}$ for the convex body defined in (6.3). Combining this with Lemma 1.2 and Proposition 6.2 we deduce $r(S_t(\vartheta_{\overline{D}}); \Delta_D) \le c\,\rho(\overline{D}(t))$ for a constant $c > 0$ not depending on t. Hence

$$\lim_{t\to\mu} \frac{\mu - t}{\rho(\overline{D}(t))} = 0$$

and so by Theorem 4.16 the essential minimum function is differentiable at $\overline{D}$ with

$$\partial_{\overline{E}}\mu^{\mathrm{ess}}(\overline{D}) = \lim_{t\to\mu} \frac{(\langle \overline{D}(t)^d \rangle \cdot \overline{E})}{\mathrm{vol}(R^t(\overline{D}))} \qquad \text{for every } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}. \tag{6.11}$$

For the formula for the derivative, we first consider the case when $\overline{E}$ is toric and semipositive.

Let $\mathfrak{S} \subset \mathfrak{M}_K$ be a finite set of places such that $u_v = 0$, $\psi_{\overline{E},v} = \Psi_E$ and $\vartheta_{\overline{E},v} = 0|_{\Delta_E}$ for all $v \in \mathfrak{M}_K \setminus \mathfrak{S}$. Let $t < \mu$ and set for short $S_t = S_t(\vartheta_{\overline{D}})$. For each $v \in \mathfrak{M}_K$ let $\theta_{v,t} \colon S_t \to \mathbb{R}$ be the restriction of the concave function $\vartheta_{\overline{D},v} - \varepsilon_v t$ to this convex body, with $\varepsilon_v = 1$ if v is Archimedean and $\varepsilon_v = 0$ otherwise. Then by Proposition 6.3 we have

$$\operatorname{vol}(R^t(\overline{D})) = d! \operatorname{vol}_M(S_t) \quad \text{and} \quad (\langle \overline{D}(t)^d \rangle \cdot \overline{E}) = \sum_{v \in \mathfrak{S}} n_v \operatorname{MI}_M(\theta_{v,t}, \dots, \theta_{v,t}, \vartheta_{\overline{E},v}). \tag{6.12}$$

Choose $x_0 \in \Delta_{D,\max}$ and set $c_v = \vartheta_{\overline{D},v}(x_0) - \langle u_v, x_0 \rangle$, $v \in \mathfrak{S}$. By (6.4) and the balancing condition of $(u_v)_v$ we have

$$\sum_{v \in \mathfrak{S}} n_v \, c_v = \sum_{v \in \mathfrak{M}_K} n_v \, (\vartheta_{\overline{D},v}(x_0) - \langle u_v, x_0 \rangle) = \vartheta_{\overline{D}}(x_0) = \mu.$$

For each $v \in \mathfrak{S}$ we have $\vartheta_{\overline{D},v}(x) \leq \langle u_v, x - x_0 \rangle + \vartheta_{\overline{D},v}(x_0) = \langle u_v, x \rangle + c_v$ for all $x \in \Delta_D$ because $u_v \in \partial \vartheta_{\overline{D},v}(x_0)$. Setting $\kappa_v = \max_{x \in S_t}(\langle u_v, x \rangle + c_v - \vartheta_{\overline{D},v}(x)) \geq 0$ we have

$$\langle u_v, x \rangle + c_v - \kappa_v - \varepsilon_v t \leq \theta_{v,t}(x) \leq \langle u_v, x \rangle + c_v - \varepsilon_v t \quad \text{for every } x \in S_t. \tag{6.13}$$

From the upper bound in (6.13) and the monotonicity of the mixed integral we get

$$\operatorname{MI}_M(\theta_{v,t}, \dots, \theta_{v,t}, \vartheta_{\overline{E},v}) \leq \operatorname{MI}_M((\langle u_v, x \rangle + c_v - \varepsilon_v \, t)|_{S_t}, \dots, (\langle u_v, x \rangle + c_v - \varepsilon_v \, t)|_{S_t}, \vartheta_{\overline{E},v}).$$

By Lemma A.10 and Remark A.11, the right-hand side of this inequality can be computed as

$$-d! \operatorname{vol}(S_t) \, \psi_{\overline{E},v}(u_v) + d \, (c_v - \varepsilon_v t) \operatorname{MV}_M(S_t, \dots, S_t, \Delta_E) + \langle u_v, x_1 \rangle$$

for a point $x_1 \in M_{\mathbb{R}}$ not depending on v, because $\psi_{\overline{E},v}$ coincides with the Legendre-Fenchel dual of $\vartheta_{\overline{E},v}$ as defined in (A.7). Summing over all these places, we deduce from (6.12) and the balancing condition of $(u_v)_v$

$$\frac{(\langle \overline{D}(t)^d \rangle \cdot \overline{E})}{\operatorname{vol}(R^t(\overline{D}))} \leq - \sum_{v \in \mathfrak{S}} n_v \psi_{\overline{E},v}(u_v) + d \, (\mu - t) \, \frac{\operatorname{MV}_M(S_t, \dots, S_t, \Delta_E)}{d! \operatorname{vol}_M(S_t)}. \tag{6.14}$$

For the converse inequality, for each $v \in \mathfrak{S}$ and any $x \in S_t$ we have

$$\mu - t \geq \mu - \vartheta_{\overline{D}}(x) = \sum_{w \in \mathfrak{S}} n_w (c_w + \langle u_w, x \rangle - \vartheta_{\overline{D},w}(x)) \geq n_v (c_v + \langle u_v, x \rangle - \vartheta_{\overline{D},v}(x))$$

using again the balancing condition of $(u_v)_v$ together with the previous upper bound for the w-adic roof functions for $w \neq v$. Since this holds for every $x \in S_t$ we deduce $n_v \kappa_v \leq \mu - t$, and in particular

$$\sum_{v \in \mathfrak{S}} n_v \kappa_v \leq \# \mathfrak{S} \, (\mu - t).$$

Combining this with the lower bound in (6.13) we similarly obtain

$$\frac{(\langle \overline{D}(t)^d \rangle \cdot \overline{E})}{\operatorname{vol}(R^t(\overline{D}))} \geq - \sum_{v \in \mathfrak{S}} n_v \psi_{\overline{E},v}(u_v) - (\# \mathfrak{S} - 1) \, d \, (\mu - t) \, \frac{\operatorname{MV}_M(S_t, \dots, S_t, \Delta_E)}{d! \operatorname{vol}_M(S_t)}. \tag{6.15}$$

Now choose $c > 0$ such that $x + c \, \Delta_E \subset \Delta_D$ for some $x \in M_{\mathbb{R}}$. Then there exists $x' \in M_{\mathbb{R}}$ such that $x' + c \, r(S_t(\vartheta_{\overline{D}}); \Delta_D) \, \Delta_E \subset S_t$, and so by the monotonicity and the multilinearity of the mixed volume function we have

$$c \, r(S_t(\vartheta_{\overline{D}}); \Delta_D) \operatorname{MV}_M(S_t, \dots, S_t, \Delta_E) \leq \operatorname{MV}_M(S_t, \dots, S_t, S_t) = d! \operatorname{vol}_M(S_t).$$

Hence with the limit (6.10) we deduce that the error terms in (6.14) and (6.15) vanish as $t \to \mu$. It then follows from the expression (6.11)

$$\partial_{\overline{E}}\mu^{\mathrm{ess}}(\overline{D}) = -\sum_{v\in\mathfrak{S}} n_v \psi_{\overline{E},v}(u_v) = -\sum_{v\in\mathfrak{M}_K} n_v \psi_{\overline{E},v}(u_v). \tag{6.16}$$

By the additivity of the derivative and the metric functions, this formula readily extends to the DSP case, and by density to any toric adelic $\mathbb{R}$-divisor on X.

For an arbitrary adelic $\mathbb{R}$-divisor $\overline{E}$ on X with E toric we apply the averaging process in Definition 6.4. By the invariance of the arithmetic positive intersection numbers with respect to this process (Proposition 6.5) we deduce from (6.11) and (6.16)

$$\partial_{\overline{E}}\mu^{\mathrm{ess}}(\overline{D}) = \partial_{\overline{E}^{\mathrm{tor}}}\mu^{\mathrm{ess}}(\overline{D}) = \sum_{v\in\mathfrak{M}_K} n_v\, \widehat{g}_v(x_v) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\eta_{v,u_v}$$

with $\widehat{g}_v$ as in Definition 6.4 and any $x_v \in \mathrm{val}_v^{-1}(u_v) \subset X_v^{\mathrm{an}}$, using the relation between Green functions and metric functions in (6.2). This completes the proof of the first statement. The second follows readily from this and Proposition 4.6. □

Now we assume that $\vartheta_{\overline{D}}$ is wide. For every $f \in K[M]\setminus\{0\}$ we introduce the quantity

$$m_{\overline{D}}(f) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} \log|f|_v\, d\eta_{\overline{D},v} \in \mathbb{R}.$$

Recall that for each v we have that $\eta_{\overline{D},v} = \eta_{v,u_v}$ for the component $u_v \in N_{\mathbb{R}}$ of the balanced family of sup-gradients of $\vartheta_{\overline{D}}$. Taking into account the definition of these probability measures and writing $f = \sum_{m\in M} \alpha_m \chi^m$ we have

$$m_{\overline{D}}(f) = \sum_{v\in\mathfrak{M}_K^\infty} n_v \int_{\mathbb{S}_v} \log|f(t\cdot x_v)|_v\, d\eta_{v,0}(t) + \sum_{v\in\mathfrak{M}_K\setminus\mathfrak{M}_K^\infty} n_v \log\max_m(e^{-\langle u_v,m\rangle}|\alpha_m|_v), \tag{6.17}$$

where for v Archimedean we denote by x_v any point in the fiber $\mathrm{val}_v^{-1}(u_v) \subset \mathbb{T}_v^{\mathrm{an}}$, and $\eta_{v,0}$ is the Haar probability measure of $\mathbb{S}_v$. Hence this quantity is an extension of the classical logarithmic Gauss-Mahler measure of a Laurent polynomial, which in our setting corresponds to the case where $u_v = 0$ for every v.

Lemma 6.7. *The following properties hold:*

(1) for every $m \in M$ and $\alpha \in K^\times$ we have $m_{\overline{D}}(\alpha\chi^m) = 0$,

(2) for every $f \in K[M] \setminus \{0\}$ we have $m_{\overline{D}}(f) \geq 0$,

(3) for every $m \in M \setminus \{0\}$ and $\gamma \in K^\times$ we have

$$m_{\overline{D}}(\chi^m - \gamma) = \sum_{v\in\mathfrak{M}_K} n_v \max(0, \langle u_v, m\rangle + \log|\gamma|_v).$$

Proof. For (1), for each $v \in \mathfrak{M}_K$ we have $\int_{X_v^{\mathrm{an}}} \log|\alpha\chi^m|_v\, d\eta_{v,u_v} = \log|\alpha|_v - \langle u_v, m\rangle$ from the explicit expression of this local term in (6.17). Hence

$$m_{\overline{D}}(\alpha\chi^m) = \sum_{v\in\mathfrak{M}_K} n_v(\log|\alpha|_v - \langle u_v, m\rangle) = 0$$

by the product formula and the fact that $(u_v)_v$ is balanced.

For (2), choose a vertex $m \in M$ of the Newton polytope of f. For each $v \in \mathfrak{M}_K$ we have

$$\int_{X_v^{\mathrm{an}}} \log |f|_v \, d\eta_{\overline{D},v} \geq \log |\alpha_m|_v - \langle u_v, m\rangle.$$

This follows again from the expression of this term in (6.17), using in the Archimedean case the fact that the Mahler measure of a Laurent polynomial is bounded below by the absolute value of any of its vertex coefficients. Together with (1) this implies

$$m_{\overline{D}}(f) \geq \sum_{v \in \mathfrak{M}_K} n_v \left(\log |\alpha_m|_v - \langle u_v, m\rangle\right) = m_{\overline{D}}(\alpha_m \chi^m) = 0.$$

For (3), for each v we have

$$\int_{X_v^{\mathrm{an}}} \log |\chi^m - \gamma|_v \, d\eta_{\overline{D},v} = \log \max(e^{-\langle u_v, m\rangle}, |\gamma|_v) = -\langle u_v, m\rangle + \max(0, \langle u_v, m\rangle + \log |\gamma|_v)$$

using again the explicit expression (6.17) together with Jensen's formula for the Mahler measure in the Archimedean case. The statement follows by considering the weighted sum of these terms and the fact that the family $(u_v)_v$ is balanced. □

For an arbitrary $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ one can compute the corresponding derivative of the essential minimum function by reducing to the situation considered in Theorem 6.6. Since the K-algebra $\mathcal{O}_X(\mathbb{T}) = K[M]$ is factorial, for any $E \in \mathrm{Div}(X)_{\mathbb{R}}$ we can choose

$$f_E \in \mathrm{Rat}(X)_{\mathbb{R}}^{\times}$$

defining the restriction of E to the torus. By Lemma 6.7(1), the quantity $m_{\overline{D}}(f_E)$ does not depend on the choice of this equation.

Corollary 6.8. *With notations and assumptions as in Theorem 6.6,*

$$\partial_{\overline{E}} \mu^{\mathrm{ess}}(\overline{D}) = m_{\overline{D}}(f_E) + \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v} \, d\eta_{\overline{D},v} \quad \textit{for every } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

Proof. We have that $\overline{E} - \widehat{\mathrm{div}}(f_E)$ is an adelic toric $\mathbb{R}$-divisor on X whose geometric $\mathbb{R}$-divisor $E - \mathrm{div}(f_E)$ is toric. Hence by the invariance of the essential minimum with respect to linear equivalence and Theorem 6.6 we obtain

$$\begin{aligned}\partial_{\overline{E}} \mu^{\mathrm{ess}}(\overline{D}) = \partial_{\overline{E} - \widehat{\mathrm{div}}(f_E)} \mu^{\mathrm{ess}}(\overline{D}) &= \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E} - \widehat{\mathrm{div}}(f_E), v} \, d\eta_{\overline{D},v} \\ &= \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} \log |f_E|_v \, d\eta_{\overline{D},v} + \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v} \, d\eta_{\overline{D},v},\end{aligned}$$

which gives the statement. □

We also have the following converse of Theorem 6.6 in the semipositive situation.

Theorem 6.9. *If $\overline{D}$ is semipositive then the following conditions are equivalent:*

(1) $\vartheta_{\overline{D}}$ is wide,

(2) the essential minimum function is differentiable at $\overline{D}$,

(3) $\overline{D}$ satisfies the equidistribution property at every place v.

To prove it, we need the next result showing that in the semipositive toric setting it is always possible to find a sharp upper bound as that required by Proposition 5.17.

Proposition 6.10. *If $\overline{D}$ is semipositive then there exists a semipositive toric adelic $\mathbb{R}$-divisor $\overline{D}'$ over D with $\overline{D}' - \overline{D}$ pseudo-effective and $\mu^{\mathrm{ess}}(\overline{D}') = \mu^{\mathrm{abs}}(\overline{D}') = \mu^{\mathrm{ess}}(\overline{D})$.*

Proof. Choose a point $x_0 \in \Delta_{D,\max}$ and let $(u_v)_v$ be a balanced family of sup-gradients for $\vartheta_{\overline{D}}$, which always exists thanks to Proposition A.8. For each $v \in \mathfrak{M}_K$ we have

$$\vartheta_{\overline{D},v}(x) \le \langle u_v, x\rangle + c_v \quad \text{for every } x \in \Delta_D \tag{6.18}$$

with $c_v = \vartheta_{\overline{D},v}(x_0) - \langle u_v, x_0\rangle$.

Using the correspondence in [BPS14, Proposition 4.9.2(2)], set $\overline{D}'$ for the semipositive toric adelic $\mathbb{R}$-divisor over D with local roof functions equal to the affine functions in the right-hand side of (6.18). Since the family $(u_v)_v$ is balanced we have

$$\vartheta_{\overline{D}'}(x) = \sum_{v\in\mathfrak{M}_K} n_v \vartheta_{\overline{D}',v}(x) = \sum_{v\in\mathfrak{M}_K} n_v(\langle u_v, x\rangle + c_v) = c \quad \text{for every } x \in \Delta_D,$$

for the constant $c = \sum_{v\in\mathfrak{M}_K} n_v c_v \in \mathbb{R}$. This implies $\mu^{\mathrm{ess}}(\overline{D}') = \mu^{\mathrm{abs}}(\overline{D}') = c$. Since $\overline{D}$ is semipositive, the inequality (6.18) implies that $\overline{D}' - \overline{D}$ is pseudo-effective. Furthermore, by (6.4) and the balancing condition for $(u_v)_v$ we have

$$c = \sum_{v\in\mathfrak{M}_K} n_v(\vartheta_{\overline{D},v}(x_0) - \langle u_v, x_0\rangle) = \vartheta_{\overline{D}}(x_0) = \mu^{\mathrm{ess}}(\overline{D})$$

and so $\mu^{\mathrm{ess}}(\overline{D}') = \mu^{\mathrm{abs}}(\overline{D}') = \mu^{\mathrm{ess}}(\overline{D})$, as stated. □

Proof of Theorem 6.9. It is a direct consequence of Theorem 6.6 together with Propositions 5.17 and 6.10. □

Remark 6.11. The toric equidistribution theorem from [BPRS19] states that in the semipositive case, the toric adelic $\mathbb{R}$-divisor $\overline{D}$ verifies the equidistribution property at every place of K if and only if it is *monocritical*, in the sense that an associate functional on a space of adelic measures has a unique global minimum. By Proposition 4.15 in *loc.cit.*, this condition can be reformulated in simpler terms as the fact that 0 is not a vertex of $\partial\vartheta_{\overline{D}}(x_0)$ for any $x_0 \in \Delta_{D,\max}$. Proposition A.3 shows that it is also equivalent to the fact that the global roof function is wide, and so Theorem 6.9 recovers this toric equidistribution theorem.

On the other hand, Theorem 6.6 extends the sufficient condition in this theorem to the situation where $\overline{D}$ is not necessarily semipositive and strengthens its conclusion to include the differentiability of the essential minimum function.

Combining the previous results with those from Section 5.4 we reinforce the toric equidistribution property of $\overline{D}$ to include test functions with logarithmic singularities along effective divisors satisfying a numerical condition.

Theorem 6.12. *Assume that $\vartheta_{\overline{D}}$ is wide and let E be an effective divisor on X such that $\mathrm{m}_{\overline{D}}(f_E) = 0$. Then for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ and $v \in \mathfrak{M}_K$ we have*

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \varphi \, d\eta_{\overline{D},v}$$

for any function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ with at most logarithmic singularities along E.

In particular, this holds if each irreducible component of the Weil divisor $[E]$ is either contained in $X \setminus \mathbb{T}$ or is the closure of the zero set of an irreducible binomial $\chi^m - \gamma$ with $m \in M \setminus \{0\}$ and $\gamma \in K^\times$ such that $\log|\gamma|_v = -\langle u_v, m\rangle$ for every v.

Proof. Since $\vartheta_{\overline{D}}$ is wide we have that $\overline{D}$ satisfies the condition in Theorem 4.16 and *a fortiori* that in Theorem 4.8. The first statement is then a direct application of Corollary 6.8 and Theorem 5.13.

For the second, in the current situation we can choose $f_E = \prod_{i \in I} f_i^{k_i}$ with $k_i \in \mathbb{N}$ and $f_i = \chi^{m_i} - \gamma_i$ for some $m_i \in M \setminus \{0\}$ and $\gamma_i \in K^\times$ such that $\log |\gamma_i|_v = -\langle u_v, m_i \rangle$ for every $i \in I$ and $v \in \mathfrak{M}_K$. By Lemma 6.7(3) we have

$$m_{\overline{D}}(f) = \sum_{i \in I} k_i \, m_{\overline{D}}(f_i) = 0,$$

so this statement follows from the first. □

It is natural to try to interpret in terms of heights the numerical condition imposed on the effective divisor E by the previous theorem. To this end, first note that for every point $x \in \mathbb{T}(\overline{K})$ we have $h_{\overline{D}}(x) \geq \mu^{\mathrm{ess}}(\overline{D})$ [BPS15, Lemma 3.8(1)], and so for every subvariety $V \subset X$ that is not contained in the boundary $X \setminus \mathbb{T}$ we have

$$\mu^{\mathrm{ess}}(\overline{D}|_V) \geq \mu^{\mathrm{ess}}(\overline{D}).$$

Following [BPRS19, Definition 5.10], we say that V is *$\overline{D}$-special* if this lower bound is an equality.

Using the characterization of the Bogomolov property for monocritical semipositive toric adelic $\mathbb{R}$-divisors in [BPRS19, Section 5] we derive the following logarithmic equidistribution theorem for the semipositive case.

Corollary 6.13. *Assume that $\overline{D}$ is semipositive and that $\vartheta_{\overline{D}}$ is wide. Let E be an effective divisor on X such that each of the irreducible components of $[E]$ is either contained in $X \setminus \mathbb{T}$ or is the closure of a $\overline{D}$-special hypersurface of $\mathbb{T}$. Then for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ and $v \in \mathfrak{M}_K$ we have*

$$\lim_{\ell \to \infty} \int_{X_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \varphi \, d\eta_{\overline{D},v}$$

for any function $\varphi \colon X_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ with at most logarithmic singularities along E.

Proof. Since $\overline{D}$ is semipositive and $\vartheta_{\overline{D}}$ is wide we have that $\overline{D}$ is monocritical in the sense of [BPRS19], see Remark 6.11. Let V be an irreducible component of $[E]$ that is $\overline{D}$-special. After extending K is necessary we assume that V is geometrically irreducible. By the Bogomolov property for monocritical adelic $\mathbb{R}$-divisors [BPRS19, Theorem 5.12], $V_0 = V \cap \mathbb{T}$ is the translate of a subtorus. Since V_0 is a hypersurface, there exist $m \in M$ and $x_0 \in \mathbb{T}(K)$ such that

$$V_0 = Z(\chi^m - 1) \cdot x_0.$$

Note that $V_0 = Z(\chi^m - \gamma)$ for $\gamma = \chi^m(x_0)$. By [BPRS19, Proposition 5.14(1)], the fact that V is $\overline{D}$-special implies that

$$u_v \in m_{\mathbb{R}}^{\perp} + \mathrm{val}_v(x_0) \quad \text{for every } v \in \mathfrak{M}_K,$$

which is equivalent to the fact that $\langle u_v, m \rangle = \langle \mathrm{val}_v(x_0), m \rangle = -\log |\gamma|_v$ for every v. We conclude with Theorem 6.12. □

7. Dynamical systems and semiabelian varieties

In this section we study adelic $\mathbb{R}$-divisors that are sums of several canonical adelic $\mathbb{R}$-divisors with different regimes with respect to an algebraic dynamical system. In this setting Zhang's lower bound for the essential minimum might be strict, in which case Yuan's equidistribution theorem cannot be applied. We show that in spite of this, the essential minimum function is differentiable at these adelic $\mathbb{R}$-divisors, and for every place the Galois orbits of small generic sequences of algebraic points converge towards the equilibrium measure (Theorem 7.4). We also show that this convergence still holds with respect to test functions with logarithmic singularities along hypersurfaces containing a dense subset of preperiodic points (Theorem 7.9).

These results apply in the setting of semiabelian varieties, giving the differentiability of the essential minimum function and recovering Kühne's semiabelian equidistribution theorem (Theorem 7.10). They also imply that this equidistribution also holds with respect to functions with logarithmic singularities along torsion hypersurfaces (Theorem 7.14).

7.1. **Canonical adelic $\mathbb{R}$-divisors.**

Canonical metrized line bundles for algebraic dynamical systems were introduced by Zhang [Zha95b] and extended to adelic $\mathbb{R}$-divisors by Chen and Moriwaki [CM15]. Here we recall this notion and study some of its positivity properties.

Let X be a normal projective variety over K and $\phi\colon X \to X$ a surjective endomorphism. Then ϕ is finite [Fak03, Lemma 5.6] and we denote by $\deg(\phi)$ its degree. Let D be an $\mathbb{R}$-divisor on X such that $\phi^*D \equiv qD$ for a real number $q > 1$.

Definition 7.1. The *canonical* adelic $\mathbb{R}$-divisor of D, denoted by $\overline{D}^{\mathrm{can}}$, is any adelic $\mathbb{R}$-divisor on X such that

$$\phi^*\overline{D}^{\mathrm{can}} \equiv q\overline{D}^{\mathrm{can}} \quad \text{on } \widehat{\mathrm{Div}}(X)_{\mathbb{R}}. \tag{7.1}$$

To construct it, choose $f \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ such that $\phi^*D = qD + \mathrm{div}(f)$. Starting from any adelic $\mathbb{R}$-divisor over D and applying Tate's limit argument, it can be shown that there exists a unique $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ such that [CM15, Section 4]

$$\phi^*\overline{D} = q\overline{D} + \widehat{\mathrm{div}}(f). \tag{7.2}$$

In particular $\overline{D}$ is canonical in the sense of Definition 7.1. Now if $\overline{D}' \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ is another canonical adelic $\mathbb{R}$-divisor over D then $\phi^*\overline{D}' = q\overline{D}' + \widehat{\mathrm{div}}(f')$ with $f' \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$. Necessarily $f = \gamma f'$ with $\gamma \in K^{\times}_{\mathbb{R}}$, and from the uniqueness of (7.2) we get $\overline{D}' = \overline{D} - \widehat{\mathrm{div}}(\lambda)$ with $\lambda = \gamma^{1/(q-1)}$. Hence the canonical adelic $\mathbb{R}$-divisor of D exists and is unique up to a summand of the form $\widehat{\mathrm{div}}(\lambda)$ with $\lambda \in K^{\times}_{\mathbb{R}}$.

The associated height function is not affected by this indeterminacy thanks to the product formula, and by (7.1) it verifies

$$h_{\overline{D}^{\mathrm{can}}}(\phi(x)) = q\, h_{\overline{D}^{\mathrm{can}}}(x) \quad \text{for every } x \in X(\overline{K}). \tag{7.3}$$

A point $x \in X(\overline{K})$ is *preperiodic* if its orbit with respect to ϕ is finite or equivalently, if there are positive integers $j < k$ such that $\phi^{\circ j}(x) = \phi^{\circ k}(x)$. The functoriality (7.3) implies that $h_{\overline{D}^{\mathrm{can}}}(x) = 0$ whenever x is preperiodic.

It is well-known that if D is ample then $\overline{D}^{\mathrm{can}}$ is nef and both the absolute and the essential minima vanish. Here we give a weaker condition ensuring the pseudo-effectivity of the canonical adelic $\mathbb{R}$-divisor and the vanishing of its essential minimum.

Proposition 7.2. *If $R(D) \neq \{0\}$ then $\overline{D}^{\mathrm{can}}$ is pseudo-effective and $\mu^{\mathrm{ess}}(\overline{D}^{\mathrm{can}}) = 0$.*

Proof. First note that the essential minimum is finite because $R(D) \neq \{0\}$. We have

$$\mu^{\mathrm{ess}}(\overline{D}^{\mathrm{can}}) = \mu^{\mathrm{ess}}(\phi^*\overline{D}^{\mathrm{can}}) = \mu^{\mathrm{ess}}(q\overline{D}^{\mathrm{can}}) = q\, \mu^{\mathrm{ess}}(\overline{D}^{\mathrm{can}})$$

since ϕ is dominant and finite, and $\phi^*\overline{D}^{\mathrm{can}} \equiv q\overline{D}^{\mathrm{can}}$. Hence $\mu^{\mathrm{ess}}(\overline{D}^{\mathrm{can}}) = 0$, as stated.

The pseudo-effectivity of $\overline{D}^{\mathrm{can}}$ then follows from Theorem 2.17 and the fact that this condition is closed. Nevertheless we give a self-contained proof of this statement.

Let $s = (f, eD)$ be a nonzero global section of eD for an integer $e \geq 1$, which exists because $R(D) \neq \{0\}$. Up to multiplying s by a nonzero scalar we can suppose that $\|s\|_{\overline{D},v,\sup} \leq 1$ for every non-Archimedean v. Then given $\varepsilon > 0$ we take $k \geq 1$ such that $\log \|s\|_{\overline{D},v,\sup} \leq \varepsilon\, e\, q^k$ for every Archimedean v. The pullback $\phi^{\circ k,*}s = (\phi^{\circ k,*}f, e\,\phi^{\circ k,*}D)$ is a nonzero global section of $e\,\phi^{\circ k,*}D$ and since ϕ is surjective, it has the same v-adic sup-norms as s. Since $\phi^*\overline{D}^{\mathrm{can}} \equiv q\,\overline{D}^{\mathrm{can}}$ there is a nonzero global section s_k of $e\,q^k D$ with the same v-adic sup-norms. Hence

$$\log \|s_k\|_{eq^k\overline{D}^{\mathrm{can}},v,\sup} = \log \|s\|_{\overline{D}^{\mathrm{can}},v,\sup} \leq \begin{cases} \varepsilon\, e\, q^k & \text{if } v \in \mathfrak{M}_K^\infty, \\ 0 & \text{if } v \in \mathfrak{M}_K \setminus \mathfrak{M}_K^\infty, \end{cases}$$

and so $s_k \in R^{-\varepsilon}(\overline{D}^{\mathrm{can}})$. Therefore $R^{-\varepsilon}(\overline{D}^{\mathrm{can}}) \neq \{0\}$ for every $\varepsilon > 0$ and so $\overline{D}^{\mathrm{can}}$ is pseudo-effective. □

7.2. **Equidistribution for sums of canonical adelic $\mathbb{R}$-divisors.** Let ϕ be a surjective endomorphism of a normal projective variety X over K of dimension $d \geq 1$. For $i = 1, \ldots, s$ let $D_i \in \mathrm{Div}(X)_{\mathbb{R}}$ with $\phi^*D_i \equiv q_i D_i$ for a real number $q_i > 1$ and set

$$\overline{D} = \sum_{i=1}^{s} \overline{D}_i^{\mathrm{can}}.$$

Up to reordering we assume that $1 < q_1 \leq q_2 \leq \cdots \leq q_s$. We also assume that

(1) $R(D_i) \neq \{0\}$ for every i,

(2) D is big and semiample.

When D is ample and D_i is nef for every i we have that $\phi^*D - D$ is ample, which by Fakhruddin's theorem [Fak03, Theorem 5.1] ensures that the set of periodic points of ϕ is dense. Together with Proposition 7.2 this easily implies that the essential minimum of $\overline{D}$ vanishes. The next result shows that this property also holds in our more general setting.

Proposition 7.3. *We have $\mu^{\mathrm{ess}}(\overline{D}) = 0$.*

Proof. By Proposition 7.2, the fact that $R(D_i) \neq \{0\}$ implies that $\mu^{\mathrm{ess}}(\overline{D}_i^{\mathrm{can}}) = 0$ for every i. Hence by Lemma 2.16(1) we have $\mu^{\mathrm{ess}}(\overline{D}) \geq \sum_{i=1}^{s} \mu^{\mathrm{ess}}(\overline{D}_i^{\mathrm{can}}) = 0$. On the other hand,

$$\phi^*\overline{D} \equiv \sum_{i=1}^{s} q_i \overline{D}_i^{\mathrm{can}} = q_1\overline{D} + \sum_{i=2}^{s} (q_i - q_1)\overline{D}_i^{\mathrm{can}}.$$

Since $q_i \geq q_1$ and $\mu^{\mathrm{ess}}(\overline{D}_i^{\mathrm{can}}) = 0$ for every i, applying again Lemma 2.16(1) and the fact that ϕ is a finite morphism we obtain $\mu^{\mathrm{ess}}(\overline{D}) = \mu^{\mathrm{ess}}(\phi^*\overline{D}) \geq q_1\mu^{\mathrm{ess}}(\overline{D})$. Hence $\mu^{\mathrm{ess}}(\overline{D}) \leq 0$, which gives the statement. □

Since ϕ is finite, for every $v \in \mathfrak{M}_K$ and any measure ν on X_v^{an} we can consider the pullback $\phi_v^{\mathrm{an},*}\nu$ by the v-adic analytification of ϕ, as explained in [Cha06, Section 2.8]. For any $\overline{A}_1, \dots, \overline{A}_d \in \widehat{\mathrm{DSP}}(X)_{\mathbb{R}}$ we have

$$\phi_v^{\mathrm{an},*}(c_1(\overline{A}_{1,v}) \wedge \cdots \wedge c_1(\overline{A}_{d,v})) = c_1(\phi^*\overline{A}_{1,v}) \wedge \cdots \wedge c_1(\phi^*\overline{A}_{d,v}). \tag{7.4}$$

The following is the central result of this section. To state it, we denote by $\mathbb{N}_d^s \subset \mathbb{N}^s$ the set of s-tuples of nonnegative integers whose components sum up to d. For each $a \in \mathbb{N}_d^s$ we set $q^a = \prod_{i=1}^s q_i^{a_i}$ and consider the subset

$$I = \{a \in \mathbb{N}_d^s \mid q^a = \deg(\phi)\}.$$

Note that there is always a semipositive adelic $\mathbb{R}$-divisor $\overline{D}'$ over D by semiampleness.

Theorem 7.4. *Let $v \in \mathfrak{M}_K$.*

(1) The essential minimum function is differentiable at $\overline{D}$, and for any semipositive adelic $\mathbb{R}$-divisor $\overline{D}'$ over D we have

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \frac{1}{(D^d)} \lim_{n\to\infty} \frac{((\phi^{\circ n,*}\overline{D}')^d \cdot \overline{E})}{\deg(\phi)^n} \quad \text{for every } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

In particular, $\overline{D}$ has the equidistribution property at v with

$$\nu_{\overline{D},v} = \frac{1}{(D^d)} \lim_{n\to\infty} \frac{\phi_v^{\circ n,\mathrm{an},*} c_1(\overline{D}_v')^{\wedge d}}{\deg(\phi)^n}.$$

(2) If each $\overline{D}_i^{\mathrm{can}}$ is DSP and $\overline{D}$ is semipositive then

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \frac{\sum_{a\in I} \binom{d}{a}(\overline{E} \cdot \prod_{i=1}^s (\overline{D}_i^{\mathrm{can}})^{a_i})}{\sum_{a\in I}\binom{d}{a}(\prod_{i=1}^s D_i^{a_i})} \quad \text{for every } \overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}.$$

(3) If each $g_{\overline{D}_i^{\mathrm{can}},v}$ is semipositive then $\nu_{\overline{D},v} = c_1(\overline{D}_v)^{\wedge d}/(D^d)$.

The next lemma gives the specific sequence of semipositive approximations that will allow us to deduce this result from Theorem 4.8.

Lemma 7.5. *Let $\overline{D}'$ be a semipositive adelic $\mathbb{R}$-divisor over D such that $\overline{D} - \overline{D}'$ is pseudo-effective, and for every $n \in \mathbb{N}$ set $\overline{Q}_n = q_s^{-n}\, \phi^{\circ n,*}\overline{D}' \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Then*

(1) $\overline{Q}_n$ is a semipositive approximation of $\overline{D}$,

(2) $r(Q_n; D) \geq q_s^{-n} q_1^n$,

(3) $(Q_n^d) = q_s^{-dn} \deg(\phi)^n (D^d)$,

(4) $\mu^{\mathrm{abs}}(\overline{Q}_n) = q_s^{-n} \mu^{\mathrm{abs}}(\overline{D}')$.

Proof. We have that $\overline{Q}_n$ is semipositive and Q_n is big because these properties are preserved under pullback with respect to a finite morphism. Set $\overline{F}_n = \phi^{\circ n,*}(\overline{D} - \overline{D}')$, which is pseudo-effective since so is $\overline{D} - \overline{D}'$. We have

$$\overline{D} - \overline{Q}_n = \overline{D} - \frac{1}{q_s^n}\phi^{\circ n,*}\overline{D}' = \overline{D} - \frac{1}{q_s^n}\phi^{\circ n,*}\overline{D} + \frac{1}{q_s^n}\overline{F}_n \equiv \sum_{i=1}^s \Big(1 - \frac{q_i^n}{q_s^n}\Big)\overline{D}_i^{\mathrm{can}} + \frac{1}{q_s^n}\overline{F}_n.$$

Hence $\overline{D} - \overline{Q}_n$ is pseudo-effective, because $q_s \geq q_i$ and by Proposition 7.2 we have that $\overline{D}_i^{\mathrm{can}}$ is pseudo-effective for every i. Thus $\overline{Q}_n$ is a semipositive approximation of $\overline{D}$, proving (1). Moreover

$$Q_n - \frac{q_1^n}{q_s^n} D \equiv \sum_{i=1}^{s} \frac{q_i^n - q_1^n}{q_s^n} D_i$$

is pseudo-effective and therefore $r(Q_n; D) \geq q_s^{-n} q_1^n$, as stated in (2). Finally the formulae (3) and (4) are respectively given by the projection formula [Ful98, Chapter 2, Proposition 2.3(c)] and the invariance of the absolute minimum with respect to pullback by a surjective morphism. □

We also need the next auxiliary result.

Lemma 7.6. *Let $I = \{a \in \mathbb{N}_d^s \mid q^a = \deg(\phi)\}$ as before.*

(1) We have $(D^d) = \sum_{a \in I} \binom{d}{a} \Big(\prod_{i=1}^{s} D_i^{a_i} \Big)$.

(2) Let $v \in \mathfrak{M}_K$ and assume that $g_{\overline{D}_i^{\mathrm{can}}, v}$ is semipositive for every i. Then

$$c_1(\overline{D}_v)^{\wedge d} = \sum_{a \in I} \binom{d}{a} \bigwedge_{i=1}^{s} c_1(\overline{D}_{i,v}^{\mathrm{can}})^{\wedge a_i} \quad \text{and} \quad \phi_v^{\mathrm{an},*} c_1(\overline{D}_v)^{\wedge d} = \deg(\phi)\, c_1(\overline{D}_v)^{\wedge d}.$$

Proof. For each $a \in \mathbb{N}_d^s$ we have

$$q^a \Big(\prod_{i=1}^{s} D_i^{a_i} \Big) = \Big(\prod_{i=1}^{s} (\phi^* D_i)^{a_i} \Big) = \deg(\phi) \Big(\prod_{i=1}^{s} D_i^{a_i} \Big)$$

by the projection formula. Therefore this quantity vanishes unless $a \in I$, and (1) follows by the multilinearity of the intersection product.

Now assume that $g_{\overline{D}_i^{\mathrm{can}}, v}$ is semipositive for every i. The multilinearity of the Monge-Ampère operator gives

$$c_1(\overline{D}_v)^{\wedge d} = \sum_{a \in \mathbb{N}_d^s} \binom{d}{a} \bigwedge_{i=1}^{s} c_1(\overline{D}_{i,v}^{\mathrm{can}})^{\wedge a_i}.$$

By semipositivity, $\bigwedge_{i=1}^{s} c_1(\overline{D}_{i,v}^{\mathrm{can}})^{\wedge a_i}$ is a measure for each $a \in \mathbb{N}_d^s$. Since its total mass is $(\prod_{i=1}^{s} D_i^{a_i})$, this measure is zero unless $a \in I$. This gives the first formula in (2). Hence

$$\begin{aligned} \phi_v^{\mathrm{an},*} c_1(\overline{D}_v)^{\wedge d} &= \sum_{a \in I} \binom{d}{a} \bigwedge_{i=1}^{s} c_1(\phi^* \overline{D}_{i,v}^{\mathrm{can}})^{\wedge a_i} \\ &= \sum_{a \in I} \binom{d}{a} q^a \bigwedge_{i=1}^{s} c_1(\overline{D}_{i,v}^{\mathrm{can}})^{\wedge a_i} = \deg(\phi)\, c_1(\overline{D}_v)^{\wedge d} \end{aligned}$$

by the functoriality (7.4), thus giving the second formula. □

Proof of Theorem 7.4. For (1) we consider first the case where $\overline{D} - \overline{D}'$ is pseudo-effective. Then for each $n \in \mathbb{N}$ we let $\overline{Q}_n = q_s^{-n} \phi^{\circ n,*}(\overline{D}')$ be the semipositive approximation of $\overline{D}$ given by Lemma 7.5. By Lemma 2.16(4) and Proposition 7.3 we have $\mu^{\mathrm{abs}}(\overline{Q}_n) \leq \mu^{\mathrm{ess}}(\overline{Q}_n) \leq \mu^{\mathrm{ess}}(\overline{D}) = 0$, and so by Lemma 7.5

$$0 \leq \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; D)} = \frac{-\mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; D)} \leq \frac{-q_s^{-n} \mu^{\mathrm{abs}}(\overline{D}')}{q_s^{-n} q_1^n} = \frac{-\mu^{\mathrm{abs}}(\overline{D}')}{q_1^n}.$$

We also have $\mu^{\mathrm{abs}}(\overline{D}') > -\infty$ because D is semiample. We deduce that this quotient vanishes as $n \to \infty$, and so by Theorem 4.8 the essential minimum function is differentiable at $\overline{D}$ and for every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ we have

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \lim_{n\to\infty} \frac{(\overline{Q}_n^d \cdot \overline{E})}{(Q_n^d)} = \lim_{n\to\infty} \frac{q_s^{-nd}((\phi^{\circ n,*}\overline{D}')^d \cdot \overline{E})}{q_s^{-nd}\deg(\phi)^n (D^d)} = \frac{1}{(D^d)} \lim_{n\to\infty} \frac{((\phi^{\circ n,*}\overline{D}')^d \cdot \overline{E})}{\deg(\phi)^n}.$$

This proves the first part of the statement in this case.

Now let $\overline{D}'$ be any semipositive adelic $\mathbb{R}$-divisor over D. Take then $\lambda \in K^{\times}$ such that $\|\lambda\|_{\overline{D}-\overline{D}',v,\sup} \le 1$ for every non-Archimedean place v. It follows that $\lambda \in \widehat{\Gamma}(X, \overline{D} - \overline{D}' + t[\infty])$ for any sufficiently large $t \in \mathbb{R}$, and so $\overline{D} - (\overline{D}' - t\,[\infty])$ is pseudo-effective. By the previous case we have

$$\begin{aligned}\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) &= \frac{1}{(D^d)} \lim_{n\to\infty} \frac{((\phi^{\circ n,*}\overline{D}' - t\,[\infty])^d \cdot \overline{E})}{\deg(\phi)^n} \\ &= \frac{1}{(D^d)} \lim_{n\to\infty} \left(\frac{((\phi^{\circ n,*}\overline{D}')^d \cdot \overline{E})}{\deg(\phi)^n} - d\,t\, \frac{((\phi^{\circ n,*}D)^{d-1} \cdot E)}{\deg(\phi)^n} \right)\end{aligned}$$

using the formula (2.4). Since the left-hand side is independent of t, it follows that

$$\lim_{n\to\infty} \frac{((\phi^{\circ n,*}D)^{d-1} \cdot E)}{\deg(\phi)^n} = 0,$$

completing the proof of this first part. The second part is a direct consequence of this one using Proposition 4.6.

For (2) let $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Since $\overline{D}$ is semipositive we can apply (1) with $\overline{D}' = \overline{D}$. Since $\phi^{\circ n,*}\overline{D} \equiv \sum_{i=1}^s q_i^n \overline{D}_i^{\mathrm{can}}$ we obtain

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \frac{1}{(D^d)} \lim_{n\to\infty} \sum_{a\in\mathbb{N}_d^s} \binom{d}{a} \Big(\frac{q^a}{\deg(\phi)}\Big)^n \Big(\overline{E} \cdot \prod_{i=1}^s (\overline{D}_i^{\mathrm{can}})^{a_i}\Big)$$

by the multilinearity of the arithmetic intersection product. The formula follows then from the existence of this limit together with Lemma 7.6(1).

For (3) note first that $g_{\overline{D},v}$ is semipositive, being a sum of semipositive v-adic Green functions. Take a semipositive adelic $\mathbb{R}$-divisor $\overline{D}'$ be over D with $g_{\overline{D}',v} = g_{\overline{D},v}$, so that $c_1(\overline{D}_v')^{\wedge d} = c_1(\overline{D}_v)^{\wedge d}$. By Lemma 7.6(2) we have $\phi_v^{\circ n,\mathrm{an},*} c_1(\overline{D}_v)^{\wedge d} = \deg(\phi)^n c_1(\overline{D}_v)^{\wedge d}$ for every $n \in \mathbb{N}$, and so the statement follows from (1). ☐

This result allows to introduce a natural notion of equilibrium measure in our present setting.

Definition 7.7. Let $v \in \mathfrak{M}_K$, choose a semipositive adelic $\mathbb{R}$-divisor $\overline{D}'$ over D and set $\mu_v = c_1(\overline{D}_v')^{\wedge d}/(D^d)$. The *$v$-adic equilibrium measure* of ϕ with respect to D is the probability measure on X_v^{an} defined as

$$\mu_{\phi,D,v} = \lim_{n\to\infty} \frac{\phi_v^{\circ n,\mathrm{an},*}\mu_v}{\deg(\phi)^n}.$$

Theorem 7.4 ensures that this limit exists and coincides with the v-adic equidistribution measure $\nu_{\overline{D},v}$. In particular it does not depend on the choice of $\overline{D}'$. By construction, the v-adic equilibrium measure is fully invariant in the sense that

$$\phi_v^{\mathrm{an},*}\mu_{\phi,D,v} = \deg(\phi)\,\mu_{\phi,D,v}.$$

Remark 7.8. When D is ample and D_i is nef for every i , the preperiodic points of ϕ form a dense subset of $X(\overline{K})$ of points of height zero with respect to $\overline{D}$. Hence in this case the v-adic equilibrium measure does not depend on D.

We also have the following logarithmic equidistribution result. Recall that a subvariety $Y \subset X$ is *preperiodic* if there are two positive integers $j < k$ such that $\phi^{\circ j}(Y) = \phi^{\circ k}(Y)$.

Theorem 7.9. *Assume that $\overline{D}_i^{\mathrm{can}}$ is semipositive for every i. Let $(x_\ell)_\ell$ be a $\overline{D}$-small generic sequence in $X(\overline{K})$ and E an effective divisor on X such that every irreducible component of its Weil divisor $[E]$ contains a dense subset of preperiodic points. Then for every $v \in \mathfrak{M}_K$ we have*

$$\lim_{\ell\to\infty}\int_{X_v^{\mathrm{an}}}\varphi\, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}}\varphi\,\frac{c_1(\overline{D}_v)^{\wedge d}}{(D^d)}$$

for any function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R}\cup\{\pm\infty\}$ with at most logarithmic singularities along E. In particular, this holds when D is ample and every irreducible component of $[E]$ is preperiodic.

Proof. Note that $\overline{D}^{\mathrm{can}}$ is semipositive, being a sum of semipositive adelic $\mathbb{R}$-divisors. For every $n \in \mathbb{N}$ let $\overline{Q}_n = q_s^{-n}\phi^{\circ n,*}\overline{D}$ be the semipositive approximation of $\overline{D}$ from Lemma 7.5. By this result and Lemma 7.6(2) we have

$$\lim_{n\to\infty}\frac{\mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n;D)} = 0 \quad\text{and}\quad \frac{c_1(\overline{Q}_{n,v})^{\wedge d}}{(Q_n^d)} = \frac{c_1(\overline{D}_v)^{\wedge d}}{(D^d)} \quad \text{for every } v\in\mathfrak{M}_K. \tag{7.5}$$

Let Y be an irreducible component of $[E]$. Up to switching to linearly equivalent divisors, we can suppose that Y is not contained in the support of any of the D_i's, and so we can consider the restriction of $\overline{Q}_n$ to Y. Then

$$h_{\overline{Q}_n}(Y) \le d\,\mu^{\mathrm{ess}}(\overline{Q}_n|_Y)\,(Q_n^{d-1}\cdot Y) \le 0.$$

by Zhang's inequality (Theorem 2.18) and the fact that the set of preperiodic points of $Y(\overline{K})$ is dense. On the other hand, let A be an ample divisor on X such that $A-Y$ is pseudo-effective. Since Q_n is nef we have $(Q_n^{d-1}\cdot Y) \le (Q_n^{d-1}\cdot A) \le (Q_n^d)/r(Q_n;A)$ by the inequality (1.1) and Lemma 1.3. Using this and Lemma 2.15 we obtain

$$0 \le \frac{h_{\overline{Q}_n}(Y) - d\,\mu^{\mathrm{abs}}(\overline{Q}_n)\,(Q_n^{d-1}\cdot Y)}{(Q_n^d)} \le \frac{-d\,\mu^{\mathrm{abs}}(\overline{Q}_n)\,(Q_n^{d-1}\cdot Y)}{(Q_n^d)} \le \frac{-d\,\mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n;A)}.$$

These inequalities with the limit in (7.5) imply that this quantity vanishes as $n \to \infty$. Since this holds for every Y, the condition of Corollary 5.15 is verified and so this result gives the first statement.

Finally, assume that D is ample and let Y be a preperiodic irreducible component of $[E]$. Then there is an integer $j > 0$ such that $Y' := \phi^{\circ j}(Y)$ is periodic with period $k_0 > 0$, and so the iteration $\phi^{\circ k_0}$ induces a dynamical system on Y'. Up to linear equivalence we can restrict D to Y' and we have

$$\phi^{\circ k_0,*}D|_{Y'} - D|_{Y'} \equiv \sum_{i=1}^{s}(q_i^{k_0}-1)D_i|_{Y'} = (q_1^{k_0}-1)D|_{Y'} + \sum_{i=1}^{s}(q_i^{k_0}-q_1^{k_0})D_i|_{Y'}.$$

The semipositivity assumption implies that D_i is nef for every i. Then $\phi^{\circ k_0,*}D|_{Y'} - D|_{Y'}$ is ample, being the sum of an ample $\mathbb{R}$-divisor and a nef one. By Fakhruddin's theorem [Fak03, Theorem 5.1] the set of periodic points of $Y'(\overline{K})$ is dense, and so $Y(\overline{K})$ contains a dense subset of preperiodic points. □

7.3. **Equidistribution on semiabelian varieties.** Here we specialize the results of the previous section in the semiabelian setting. We first recall the basic constructions and properties that are needed to this end, referring to [Cha00, Küh22] for the proofs and more details.

Let G be a semiabelian variety over K that is the extension of an abelian variety A of dimension g by a split torus $\mathbb{G}_{\mathrm{m}}^r$. Hence there is an exact sequence of commutative algebraic groups over K

$$0 \longrightarrow \mathbb{G}_{\mathrm{m}}^r \longrightarrow G \longrightarrow A \longrightarrow 0.$$

We consider the compactification $\overline{G}$ of G induced by toric compactification $(\mathbb{P}^1)^r$ of $\mathbb{G}_{\mathrm{m}}^r$. To construct it, one endows the product variety $G \times (\mathbb{P}^1)^r$ with the action of this split torus defined at the level of points as

$$t \cdot (x, y) = (t \cdot_G x, t^{-1} \cdot_{(\mathbb{P}^1)^r} y)$$

and defines $\overline{G}$ as the categorical quotient $G \times (\mathbb{P}^1)^r / \mathbb{G}_{\mathrm{m}}^r$. It is a smooth variety over K containing G as a dense open subset, and the projection $G \to A$ extends to a morphism $\pi \colon \overline{G} \to A$ allowing to consider this compactification as a $(\mathbb{P}^1)^r$-bundle over A.

For a given integer $\ell > 1$ the multiplication-by-ℓ on G extends to a morphism $[\ell]_{\overline{G}} \colon \overline{G} \to \overline{G}$ of degree ℓ^{r+2g}. If we denote by $[\ell]_A$ the multiplication-by-ℓ on A, then there is a commutative diagram

$$\begin{tikzcd} \overline{G} \arrow[r, "{[\ell]_{\overline{G}}}"] \arrow[d, "\pi"] & \overline{G} \arrow[d, "\pi"] \\ A \arrow[r, "{[\ell]_A}"] & A \end{tikzcd} \tag{7.6}$$

The boundary $\overline{G} \setminus G$ is an effective Weil divisor, and we denote by M its associated (Cartier) divisor on $\overline{G}$. It is relatively ample with respect to π and verifies

$$[\ell]_{\overline{G}}^* M = \ell M \quad \text{ on } \operatorname{Div}(\overline{G}).$$

Let N be an ample symmetric divisor on A, which therefore verifies that $[\ell]_A^* N \equiv \ell^2 N$ on $\operatorname{Div}(A)$. Then its pullback $\pi^* N$ is semiample, and by (7.6) it verifies

$$[\ell]_{\overline{G}}^* \pi^* N \equiv \ell^2 \pi^* N \quad \text{ on } \operatorname{Div}(\overline{G}).$$

Furthermore, the sum $D = M + \pi^* N$ is an ample divisor on $\overline{G}$.

Let $\overline{M}^{\mathrm{can}} \in \widehat{\operatorname{Div}}(\overline{G})$ and $\overline{N}^{\mathrm{can}} \in \widehat{\operatorname{Div}}(A)$ be the canonical adelic divisors of M and N for the surjective endomorphisms $[\ell]_{\overline{G}}$ and $[\ell]_A$, respectively. By [Cha00, Proposition 3.4] the adelic divisor $\overline{M}^{\mathrm{can}}$ does not depend of the choice of ℓ, and the same holds for $\overline{N}^{\mathrm{can}}$. By the commutativity in (7.6) we have

$$[\ell]_{\overline{G}}^* \overline{M}^{\mathrm{can}} = \ell \overline{M}^{\mathrm{can}} \quad \text{and} \quad [\ell]_{\overline{G}}^* \pi^* \overline{N}^{\mathrm{can}} \equiv \ell^2 \pi^* \overline{N}^{\mathrm{can}} \quad \text{ on } \widehat{\operatorname{Div}}(\overline{G}). \tag{7.7}$$

In particular, $\pi^* \overline{N}^{\mathrm{can}}$ is the canonical adelic divisor of $\pi^* N$ for $[\ell]_{\overline{G}}$.

We have that $\overline{M}^{\mathrm{can}}$ is semipositive, as shown by Chambert-Loir in [Cha00, Proposition 3.6] relying on some specific regular models of abelian varieties constructed by

Künnemann. The adelic divisor $\overline{N}^{\mathrm{can}}$ on A is semipositive because N is ample, and so this is also the case for $\pi^*\overline{N}^{\mathrm{can}}$.

Finally set $\overline{D} = \overline{M}^{\mathrm{can}} + \pi^*\overline{N}^{\mathrm{can}} \in \widehat{\mathrm{Div}}(\overline{G})$. By (7.7), its height function verifies

$$h_{\overline{D}}([\ell]_{\overline{G}}x) = \ell\, h_{\overline{M}^{\mathrm{can}}}(x) + \ell^2\, h_{\overline{N}^{\mathrm{can}}}(\pi(x)) \quad \text{for } x \in \overline{G}(\overline{K}).$$

It is nonnegative on $G(\overline{K})$ and vanishes on the torsion points, and so $\mu^{\mathrm{ess}}(\overline{D}) = 0$. On the other hand, this height function might take negative values at the points in the boundary $\overline{G} \setminus G$ [Cha00, Corollaire 4.6]. In these cases we have $\mu^{\mathrm{abs}}(\overline{D}) < 0$ and so $\overline{D}$ is outside of the scope of Yuan's equidistribution theorem.

The next result is a direct application of Theorem 7.4.

Theorem 7.10. *The essential minimum function is differentiable at $\overline{D}$ with*

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \frac{((\overline{M}^{\mathrm{can}})^r \cdot (\pi^*\overline{N}^{\mathrm{can}})^g \cdot \overline{E})}{(M^r \cdot \pi^* N^g)} \quad \textit{for every } \overline{E} \in \widehat{\mathrm{Div}}(\overline{G})_{\mathbb{R}}.$$

In particular, $\overline{D}$ satisfies the v-adic equidistribution property at every $v \in \mathfrak{M}_K$ with

$$\nu_{\overline{D},v} = \frac{c_1(\overline{M}_v^{\mathrm{can}})^{\wedge r} \wedge c_1(\pi^*\overline{N}_v^{\mathrm{can}})^{\wedge g}}{(M^r \cdot \pi^* N^g)} = \frac{c_1(\overline{D}_v)^{\wedge r+g}}{(D^{r+g})}.$$

Proof. We have $R(M) \neq \{0\}$ because M is effective, and $R(\pi^*N) \neq \{0\}$ because π^*N is semiample. As explained, D is ample and both $\overline{M}^{\mathrm{can}}$ and $\pi^*\overline{N}^{\mathrm{can}}$ are semipositive. Then Theorem 7.4 gives the stated differentiability for the essential minimum function.

To apply the formula of Theorem 7.4(2) for the derivative $\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$ we need to determine the elements $a \in \mathbb{N}^2_{r+g}$ for which $\ell^{a_1+2a_2} = \deg([\ell]_{\overline{G}}) = \ell^{r+2g}$. The only one is $a = (r, g)$, and so we obtain the desired expression. The formulae for the v-adic equidistribution measure then follow from Proposition 4.6 and Theorem 7.4(3). □

Remark 7.11. When v is Archimedean, the equidistribution measure in this result coincides with the Haar probability measure on the maximal compact subgroup $\mathbb{S}_v \simeq (S^1)^{r+2g}$ of G_v^{an}, see for instance [Küh22, Lemma 5.2].

When v is non-Archimedean, the description of this measure seems more complicated. For abelian varieties, they were described by Gubler in terms of convex geometry [Gub10] but the extension to the semiabelian case is still pending.

Remark 7.12. In our current semiabelian setting, the sequence of semipositive approximations of $\overline{D}$ from Lemma 7.5 applied with $\overline{D}' = \overline{D}$ verifies

$$\overline{Q}_n \equiv \ell^{-n}\overline{M}^{\mathrm{can}} + \pi^*\overline{N}^{\mathrm{can}}, \quad n \in \mathbb{N},$$

and for the corresponding inradius, degree and absolute minimum we have $r(Q_n; D) \geq \ell^{-n}$, $(Q_n^{r+g}) = \ell^{-rn}(D^{r+g})$ and $\mu^{\mathrm{abs}}(\overline{Q}_n) = \ell^{-2n}\mu^{\mathrm{abs}}(\overline{D})$ for each n. Hence

$$0 \leq \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; D)} = \frac{-\mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; D)} \leq \frac{-\mu^{\mathrm{abs}}(\overline{D})}{\ell^n}$$

and so the condition (4.2) is satisfied. On the other hand, this is not the case for the stronger condition from Remark 4.9 as soon as $r \geq 2$.

We next extend this equidistribution result to the closure of a subvariety of G with vanishing essential minimum. By the semiabelian Bogomolov conjecture, proved by David and Philippon [DP00], these subvarieties are translates of semiabelian subvarieties by torsion points, and so they do not provide examples of equidistribution phenomena beyond those already obtained. However this extension is the centerpiece of Kühne's approach to this conjecture [Küh22, Proposition 4.1] and so it is worth showing that it can also be derived from our results.

Let $Y \subset \overline{G}$ be the closure of a subvariety of G, and set $e = \dim(Y)$ and $e' = \dim(\pi(Y))$. Then Y is not contained in the support of M, and after possibly replacing the divisor $N \in \mathrm{Div}(A)$ by a linearly equivalent one, we assume without loss of generality that Y is neither contained in the support of π^*N. Hence we can consider the restriction of $\overline{D}$ to this subvariety.

Proposition 7.13. *With notation as above, assume that $\mu^{\mathrm{ess}}(\overline{D}|_Y) = 0$. Then $\overline{D}|_Y$ satisfies the v-adic equidistribution property for every $v \in \mathfrak{M}_K$ with*

$$\nu_{\overline{D}|_Y, v} = \frac{c_1(\overline{M}_v^{\mathrm{can}})^{\wedge e-e'} \wedge c_1(\pi^*\overline{N}_v^{\mathrm{can}})^{\wedge e'} \wedge \delta_{Y_v^{\mathrm{an}}}}{(M^{e-e'} \cdot \pi^*N^{e'} \cdot Y)}.$$

Proof. For each $n \in \mathbb{N}$ let $\overline{Q}_n = \ell^{-n}\overline{M}^{\mathrm{can}} + \pi^*\overline{N}^{\mathrm{can}}$. Then Q_n is ample, $\overline{Q}_n$ is semipositive and $\overline{D} - \overline{Q}_n$ is effective. Let $\widetilde{Y}$ be the normalization of the subvariety Y, and denote by $\overline{D}|_{\widetilde{Y}}$ and $\overline{Q}_n|_{\widetilde{Y}}$ the adelic $\mathbb{R}$-divisors on $\widetilde{Y}$ obtained by pullback. Since the normalization morphism is birational and Y is not contained in the support of M and π^*N, we have that $\overline{Q}_n|_{\widetilde{Y}}$ is a semipositive approximation of $\overline{D}|_{\widetilde{Y}}$. Its absolute minimum can be estimated as

$$0 = \mu^{\mathrm{ess}}(\overline{D}|_Y) = \mu^{\mathrm{ess}}(\overline{D}|_{\widetilde{Y}}) \geq \mu^{\mathrm{abs}}(\overline{Q}_n|_{\widetilde{Y}}) \geq \mu^{\mathrm{abs}}(\overline{Q}_n) = \ell^{-2n}\mu^{\mathrm{abs}}(\overline{D}),$$

where the last equality comes from the fact that $\overline{Q}_n \equiv \ell^{-2n}([\ell]_{\overline{G}}^{\circ n,*}\overline{D})$ and the invariance of the absolute minimum with respect to pullback by surjective morphisms. On the other hand we have that $Q_n|_{\widetilde{Y}} - \ell^{-n}D|_{\widetilde{Y}} = (1-\ell^{-n})\pi^*N|_{\widetilde{Y}}$ is effective and so

$$r(Q_n|_{\widetilde{Y}}; D|_{\widetilde{Y}}) \geq \ell^{-n}.$$

Hence $\lim_{n\to\infty} \mu^{\mathrm{abs}}(\overline{Q}_n|_{\widetilde{Y}})/r(Q_n|_{\widetilde{Y}}; D|_{\widetilde{Y}}) = 0$, and so Theorem 4.8 and Remark 4.10 imply that $\overline{D}|_Y$ satisfies the equidistribution property for every $v \in \mathfrak{M}_K$ with

$$\nu_{\overline{D}|_Y, v} = \lim_{n\to\infty} \frac{c_1((\overline{Q}_n|_Y)_v)^{\wedge e}}{((Q_n|_Y)^e)} = \lim_{n\to\infty} \frac{c_1(\overline{Q}_{n,v})^{\wedge e} \wedge \delta_{Y_v^{\mathrm{an}}}}{(Q_n^e \cdot Y)}.$$

To compute this limit, first note that

$$(Q_n^e \cdot Y) = \sum_{j=0}^{e} \ell^{-n(e-j)} \binom{e}{j} (M^{e-j} \cdot \pi^*N^j \cdot Y).$$

For each j consider the intersection product $[M^{e-j} \cdot Y]$ in the Chow group of j-dimensional cycles of Y. By the projection formula we have

$$(M^{e-j} \cdot \pi^*N^j \cdot Y) = (\pi_*[M^{e-j} \cdot Y] \cdot N^j),$$

where the left intersection number is computed over Y and the right over $\pi(Y)$. In particular this quantity vanishes for $j > e' = \dim(\pi(Y))$. On the other hand, for $j = e'$ it is equal to $(M^{e-e'} \cdot F)\,(N^{e'})$ for a general fiber F of the projection $Y \to \pi(Y)$, and therefore it is positive because M is relatively ample and N is ample. Hence

$$(Q_n^e \cdot Y) = \ell^{-n(e-e')} \binom{e}{e'} (M^{e-e'} \cdot \pi^*N^{e'} \cdot Y) + O(\ell^{-n(e-e'+1)}), \tag{7.8}$$

and the dominant term in this asymptotics is positive.

Furthermore the measure $c_1(\overline{Q}_{n,v})^{\wedge e} \wedge \delta_{Y_v^{\mathrm{an}}}$ is zero whenever $j > e'$ because its total mass vanishes, and therefore

$$c_1(\overline{Q}_{n,v})^{\wedge e} \wedge \delta_{Y_v^{\mathrm{an}}} = \sum_{j=0}^{e'} \ell^{-n(e-j)} \binom{e}{j} c_1(\overline{M}_v^{\mathrm{can}})^{\wedge e-j} \wedge c_1(\pi^* \overline{N}_v^{\mathrm{can}})^{\wedge j} \wedge \delta_{Y_v^{\mathrm{an}}}.$$

The statement then follows by taking the limit for $n \to \infty$ of the ratio between this asymptotics and that in (7.8). □

Finally we strengthen the semiabelian equidistribution property to include test functions with logarithmic singularities along the closure of a torsion hypersurface or an irreducible component of the boundary. Recall that a hypersurface of G is *torsion* if it is the translate of a semiabelian hypersurface of G by a torsion point.

Theorem 7.14. *Let $(x_\ell)_\ell$ be a $\overline{D}$-small generic sequence in $\overline{G}(\overline{K})$ and E an effective divisor on $\overline{G}$ such that each irreducible component of $[E]$ is either the closure of a torsion hypersurface of G or an irreducible component of $\overline{G} \setminus G$. Then for every $v \in \mathfrak{M}_K$ and any function $\varphi\colon \overline{G}_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ with at most logarithmic singularities along E we have*

$$\lim_{\ell \to \infty} \int_{\overline{G}_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \int_{\overline{G}_v^{\mathrm{an}}} \varphi \, \frac{c_1(\overline{D}_v)^{\wedge r+g}}{(D^{r+g})}.$$

Proof. This follows from Theorem 7.9 noting that every irreducible component of $[E]$ is a preperiodic hypersurface for the endomorphism $[\ell]_{\overline{G}}$ for any $\ell > 1$. □

8. Quasi-projective varieties

In this section we extend our study to the setting of adelic line bundles on quasi-projective varieties in the sense of Yuan and Zhang [YZ26]. We start by recalling the elements of this theory in terms of adelic $\mathbb{R}$-divisors on quasi-projective varieties, following the presentation of Burgos and Kramer in [BK25, Section 3]. Once the basic constructions and facts are achieved, our arguments can be applied in a rather direct way. For brevity we focus on the variant from Section 5.3, whose extension (Theorem 8.11) generalizes Yuan and Zhang's quasi-projective equidistribution theorem.

8.1. **Adelic $\mathbb{R}$-divisors on quasi-projective varieties.** First we consider the geometric case. Let X be a normal projective variety over K of dimension $d \geq 1$ and B an effective divisor on X. Set $U = X \setminus \mathrm{supp}(B)$ and let $R(X,U)$ be the category of normal modifications of X which are isomorphisms over U. Given such a modification $\pi\colon X_\pi \to X$, we write (X_π, π) or simply π for the corresponding object in $R(X,U)$. The space of *model* $\mathbb{R}$-*divisors* on U is defined as the direct limit

$$\mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{mod}} = \varinjlim_{\pi \in R(X,U)} \mathrm{Div}(X_\pi)_{\mathbb{R}}.$$

Given $D, D' \in \mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{mod}}$ we write $D \geq D'$ or $D' \leq D$ if there exists $(X_\pi, \pi) \in R(X,U)$ such that $D, D' \in \mathrm{Div}(X_\pi)_{\mathbb{R}}$ and $D - D'$ is effective. The B-adic norm on $\mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{mod}}$ (with possibly infinite values) is defined as

$$\|D\|_B = \inf\{\varepsilon \in \mathbb{R}_{>0} \mid -\varepsilon B \leq D \leq \varepsilon B\} \quad \text{for every } D \in \mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{mod}}.$$

The space $\mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{adel}}$ is then defined as the completion of $\mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{mod}}$ for the B-adic topology, and its elements are called *adelic* $\mathbb{R}$-*divisors* on U. This space depends on the open subset U but not on the effective divisor B.

Let $D \in \mathrm{Div}(U)^{\mathrm{adel}}_{\mathbb{R}}$ and $(D_i)_i$ a Cauchy sequence in $\mathrm{Div}(U)^{\mathrm{mod}}_{\mathbb{R}}$ representing this adelic $\mathbb{R}$-divisor. The *volume* of D is defined as

$$\mathrm{vol}(D) = \lim_{i\to\infty} \mathrm{vol}(D_i).$$

It follows from [YZ26, Theorems 5.2.1 and 5.2.9] that this limit exists in $\mathbb{R}$ and does not depend on the choice of the approximating sequence.

We say that D is *big* if $\mathrm{vol}(D) > 0$. We also say that D is *nef* if the sequence $(D_i)_i$ can be chosen such that D_i is nef for every i. We then say that D is *integrable* if it can be written as $D = A_1 - A_2$ with $A_1, A_2 \in \mathrm{Div}(U)^{\mathrm{adel}}_{\mathbb{R}}$ nef. The subspace of integrable adelic $\mathbb{R}$-divisors on U is denoted by $\mathrm{Div}(U)^{\mathrm{int}}_{\mathbb{R}}$.

The *intersection product* of integrable adelic $\mathbb{R}$-divisors is the symmetric multilinear map from [BK25, Theorem 3.7]

$$(D_1, \dots, D_d) \in (\mathrm{Div}(U)^{\mathrm{int}}_{\mathbb{R}})^d \longmapsto (D_1 \cdots D_d) \in \mathbb{R}.$$

For $j = 1, \dots, d$ let D_j be a nef adelic $\mathbb{R}$-divisor on U and choose a Cauchy sequence $(D_{j,i})_i$ of nef model $\mathbb{R}$-divisors on U representing D_j. Then

$$(D_1 \cdots D_d) = \lim_{i\to\infty} (D_{1,i} \cdots D_{d,i})$$

with the intersection products in the right-hand side computed in common models. By [BK25, Theorem 3.7], this limit exists in $\mathbb{R}$ and does not depend on the choice of the sequences. We have $(D^d) = \mathrm{vol}(D)$ for every nef $D \in \mathrm{Div}(U)^{\mathrm{adel}}_{\mathbb{R}}$.

Definition 8.1. Let P, A be big adelic $\mathbb{R}$-divisors on U. The *inradius* of P with respect to A is defined as

$$r(P; A) = \sup\{\lambda \in \mathbb{R} \mid P - \lambda A \text{ is big}\}.$$

Lemma 8.2. *Let P, A be big adelic $\mathbb{R}$-divisors on U, and let $(P_i)_i$ and $(A_i)_i$ be Cauchy sequences in $\mathrm{Div}(U)^{\mathrm{mod}}$ representing P and A. Then*

$$r(P; A) = \lim_{i\to\infty} r(P_i; A_i).$$

Moreover, $r(P; A)$ is a positive real number.

Proof. By definition we have

$$\lim_{i\to\infty} \mathrm{vol}(P_i - \lambda A_i) = \mathrm{vol}(P - \lambda A) \quad \text{for every } \lambda \in \mathbb{R}. \tag{8.1}$$

Let $\lambda < r(P; A)$. Then $P - \lambda A$ is big and so the right-hand side of (8.1) is strictly positive, which implies that $P_i - \lambda A_i$ is big for every i large enough. Since λ is arbitrary, we obtain

$$\liminf_{i\to\infty} r(P_i; A_i) \geq r(P; A).$$

Now let $\lambda < \limsup_{i\to\infty} r(P_i; A_i)$. Then there are subsequences $(P_{i_k})_k$ and $(A_{i_k})_k$ and a constant $c > 0$ such that $(P_{i_k} - \lambda A_{i_k}) - cA_{i_k}$ is big for every k. Hence the left-hand side of (8.1) is strictly positive and $P - \lambda A$ is big. Since λ is arbitrary, we get

$$\limsup_{i\to\infty} r(P_i; A_i) \leq r(P; A)$$

thus completing the proof of the first statement.

For the second, since bigness is an open condition we have $r(P; A) > 0$. This condition also implies that there exist $P', A' \in \mathrm{Div}(X)_{\mathbb{R}}$ with $P' - P$ and $A - A'$ big, and so $r(P; A) \leq r(P'; A') < \infty$. □

Next we consider the arithmetic case. An *arithmetic variety* $\mathcal{X}$ over $\mathcal{O}_K$ is a flat integral scheme over $\mathrm{Spec}(\mathcal{O}_K)$. Assume that $\mathcal{X}$ is normal and projective of dimension $d+1$ and denote by X its generic fiber, which is a normal projective variety over K of dimension d. We denote by $\mathrm{Div}(\mathcal{X})_{\mathbb{R}}$ the space of $\mathbb{R}$-divisors on $\mathcal{X}$, and for $\mathcal{D} \in \mathrm{Div}(\mathcal{X})_{\mathbb{R}}$ we denote by $\mathcal{D}|_X \in \mathrm{Div}(X)_{\mathbb{R}}$ the restriction to X.

An *arithmetic* $\mathbb{R}$-*divisor* on $\mathcal{X}$ is a pair $\overline{\mathcal{D}} = (\mathcal{D}, (g_v)_{v \in \mathfrak{M}_K^\infty})$ where $\mathcal{D}$ is an $\mathbb{R}$-divisor on $\mathcal{X}$ and g_v a continuous v-adic Green function for $\mathcal{D}|_X$ for every $v \in \mathfrak{M}_K^\infty$. The space of arithmetic $\mathbb{R}$-divisors on $\mathcal{X}$ is denoted by $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{R}}$. We say that $\overline{\mathcal{D}}$ is *effective* (respectively *strictly effective*) if $\mathcal{D}$ is effective and $g_v \geq 0$ (respectively $g_v > 0$) on $X_v^{\mathrm{an}} \setminus \mathrm{supp}(D)_v^{\mathrm{an}}$ for every v.

Let $\overline{\mathcal{B}} = (\mathcal{B}, (g_{\overline{\mathcal{B}},v})_v)$ be a strictly effective arithmetic divisor on $\mathcal{X}$ and set $\mathcal{U} = \mathcal{X} \setminus \mathrm{supp}(\mathcal{B})$. Consider also the underlying divisor and the open subset

$$B = \mathcal{B}|_X \in \mathrm{Div}(X) \quad \text{and} \quad U = X \setminus \mathrm{supp}(B) \subset X.$$

We denote by $R(\mathcal{X},\mathcal{U})$ the category of normal modifications $\pi\colon \mathcal{X}_\pi \to \mathcal{X}$ that are isomorphisms over $\mathcal{U}$. Such a normal modification is denoted by $(\pi, \mathcal{X}_\pi)$ or simply by π. The space of *model arithmetic* $\mathbb{R}$-*divisors* on $\mathcal{U}$ is the direct limit

$$\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{mod}} = \varinjlim_{\pi \in R(\mathcal{X},\mathcal{U})} \widehat{\mathrm{Div}}(\mathcal{X}_\pi)_{\mathbb{R}}.$$

Given $\overline{\mathcal{D}}_1, \overline{\mathcal{D}}_2 \in \widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{mod}}$ we write $\overline{\mathcal{D}}_1 \geq \overline{\mathcal{D}}_2$ or $\overline{\mathcal{D}}_2 \leq \overline{\mathcal{D}}_1$ whenever $\overline{\mathcal{D}}_1 - \overline{\mathcal{D}}_2$ is effective on a model where both $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ are defined. The $\overline{\mathcal{B}}$-adic norm is defined as

$$\|\overline{\mathcal{D}}\|_{\overline{\mathcal{B}}} = \inf\{\varepsilon \in \mathbb{R}_{>0} \mid -\varepsilon\overline{\mathcal{B}} \leq \overline{\mathcal{D}} \leq \varepsilon\overline{\mathcal{B}}\} \quad \text{for } \overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{mod}}.$$

The space $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{adel}}$ of *adelic* $\mathbb{R}$-*divisors* on $\mathcal{U}$ is then defined as the completion of $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{mod}}$ with respect to the $\overline{\mathcal{B}}$-adic topology. As in the geometric case, it depends only on the open subset $\mathcal{U}$.

Remark 8.3. In [BK25, Section 3.3] arithmetic varieties are required to have smooth generic fiber, and arithmetic $\mathbb{R}$-divisors are assumed to be of smooth type. Nevertheless, our space $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{adel}}$ coincides with that in *loc. cit.* up to possibly shrinking the open subset $\mathcal{U}$, by the existence of compactifications with smooth generic fiber and the density of Green functions of smooth type among those of continuous type [BK25, Remark 3.14].

Given $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{adel}}$ we denote by $(\overline{\mathcal{D}}_i)_i$ a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{mod}}$ representing this adelic $\mathbb{R}$-divisor. For convenience, we assume that such sequences have bounded differences even for small indices, namely that there exists $c \in \mathbb{R}_{\geq 0}$ with

$$\|\overline{\mathcal{D}}_i - \overline{\mathcal{D}}_j\|_{\overline{\mathcal{B}}} \leq c \quad \text{for every } i, j. \tag{8.2}$$

For each i choose $(\pi_i, \mathcal{X}_i) \in R(\mathcal{X},\mathcal{U})$ such that $\overline{\mathcal{D}}_i \in \widehat{\mathrm{Div}}(\mathcal{X}_i)_{\mathbb{R}}$, set X_i for the generic fiber of $\mathcal{X}_i$ and then $D_i = \mathcal{D}_i|_{X_i}$. We have that $(D_i)_i$ is a sequence in $\mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{mod}}$ that is Cauchy for the B-adic topology and so defines an element $D \in \mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{adel}}$, called the *geometric* adelic $\mathbb{R}$-divisor of $\overline{\mathcal{D}}$.

Also for each i we write $\overline{\mathcal{D}}_i = (\mathcal{D}_i, (g_{i,v})_{v \in \mathfrak{M}_K^\infty}) \in \widehat{\mathrm{Div}}(\mathcal{X}_i)_{\mathbb{R}}$ and we denote by

$$\overline{D}_i = (D_i, (g_{i,v})_{v \in \mathfrak{M}_K}) \in \widehat{\mathrm{Div}}(X_i)_{\mathbb{R}} \tag{8.3}$$

the adelic $\mathbb{R}$-divisor on X_i in the sense of Definition 2.1 obtained by adding the non-Archimedean Green functions induced by $\mathcal{D}_i$, as explained in Section 2.1.

Remark 8.4. We denote the elements of $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{R}}$ by overlined calligraphic letters and following the pattern described above, the akin elements of $\mathrm{Div}(\mathcal{X})_{\mathbb{R}}$, $\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and $\mathrm{Div}(X)_{\mathbb{R}}$ are denoted by either the same calligraphic letter or the corresponding Roman one, and either keeping or not the overline. Similarly, the elements of $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{adel}}$ are denoted by overlined calligraphic letters and those in $\mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{adel}}$ by the corresponding non-overlined Roman letter.

We say that $\overline{\mathcal{D}}$ is *pseudo-effective* (respectively *semipositive*, respectively *nef*) if the sequence $(\overline{\mathcal{D}}_i)_i$ can be chosen such that $\overline{\mathcal{D}}_i$ is pseudo-effective (respectively semipositive, respectively nef) for every i. We say that $\overline{\mathcal{D}}$ is *integrable* if it is the difference of two nef adelic $\mathbb{R}$-divisors on $\mathcal{U}$. The subspace of integrable adelic $\mathbb{R}$-divisors on $\mathcal{U}$ is denoted by $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{int}}$.

Remark 8.5. Our definition of nef adelic $\mathbb{R}$-divisors on $\mathcal{U}$ coincides with that in [BK25] but differs slightly from the one in [YZ26], where nef adelic divisors in the above sense are called strongly nef. However, the two definitions coincide after possibly shrinking the open subset $\mathcal{U}$ [BK25, Remarks 3.5, 3.16 and 3.39].

When $\overline{\mathcal{D}}$ is integrable, for each $v \in \mathfrak{M}_K$ we denote by $c_1(\overline{\mathcal{D}}_v)^{\wedge d}$ the signed measure on U_v^{an} defined in [YZ26, Section 3.6.7] and extended to integrable adelic $\mathbb{R}$-divisors by multilinearity. By [YZ26, Lemma 5.4.4] it has total mass (D^d). If $\overline{\mathcal{D}}$ is nef and $\overline{\mathcal{D}}_i$ is also nef for every i, then

$$\int_{U_v^{\mathrm{an}}} \varphi\, c_1(\overline{\mathcal{D}}_v)^{\wedge d} = \lim_{i\to\infty} \int_{X_{i,v}^{\mathrm{an}}} \varphi\, c_1(\overline{D}_{i,v})^{\wedge d} \tag{8.4}$$

for every continuous function $\varphi\colon U_v^{\mathrm{an}} \to \mathbb{R}$ with compact support. Here we view φ as a function on $X_{i,v}^{\mathrm{an}}$ via the open immersion $U_v^{\mathrm{an}} \hookrightarrow X_{i,v}^{\mathrm{an}}$.

For each i we denote by $h_{\overline{\mathcal{D}}_i}\colon X_i(\overline{K}) \to \mathbb{R}$ the height function of $\overline{D}_i \in \widehat{\mathrm{Div}}(X_i)_{\mathbb{R}}$. The fact that $(\overline{\mathcal{D}}_i)_i$ is a Cauchy sequence readily implies that $\lim_{i\to\infty} h_{\overline{\mathcal{D}}_i}(x)$ exists for every $x \in U(\overline{K})$. This limit does not depend on the choice of the sequence, and so we define the height function $h_{\overline{\mathcal{D}}}\colon U(\overline{K}) \longrightarrow \mathbb{R}$ by setting

$$h_{\overline{\mathcal{D}}}(x) = \lim_{i\to\infty} h_{\overline{\mathcal{D}}_i}(x) \quad \text{for every } x \in U(\overline{K}).$$

The arithmetic intersection product of integrable adelic $\mathbb{R}$-divisors is the symmetric multilinear map from [BK25, Theorem 3.37]

$$(\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}) \in (\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{int}})^{d+1} \longmapsto (\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_{d+1}) \in \mathbb{R}.$$

For $j = 1, \dots, d+1$ let $\overline{\mathcal{D}}_j$ be a nef adelic $\mathbb{R}$-divisor on $\mathcal{U}$ and choose a Cauchy sequence $(\overline{\mathcal{D}}_{j,i})_i$ of nef model $\mathbb{R}$-divisors on $\mathcal{U}$ representing $\overline{\mathcal{D}}_j$. Then

$$(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_{d+1}) = \lim_{i\to\infty} (\overline{D}_{1,i} \cdots \overline{D}_{d+1,i}),$$

where $\overline{D}_{i,j}$, $j = 1, \dots, d+1$, are the associated adelic $\mathbb{R}$-divisors as in (8.3) and the arithmetic intersection products in the right-hand side are computed in common models.

For any dominant morphism $\phi\colon \mathcal{U}' \to \mathcal{U}$ of normal quasi-projective arithmetic varieties there is a pullback map $\phi^*\colon \widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{adel}} \to \widehat{\mathrm{Div}}(\mathcal{U}')_{\mathbb{R}}^{\mathrm{adel}}$ [BK25, Section 3.5]. If ϕ is birational then

$$(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_{d+1}) = (\phi^*\overline{\mathcal{D}}_1 \cdots \phi^*\overline{\mathcal{D}}_{d+1}).$$

We denote by $[\infty] \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{R}}$ the arithmetic divisor over the zero divisor on $\mathcal{X}$ with $g_v = 1$ for every $v \in \mathfrak{M}_K^{\infty}$. We have

$$(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_d \cdot [\infty]) = (D_1 \cdots D_d).$$

8.2. **Essential and absolute minima.** Let $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{adel}}$ with big $D \in \mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{adel}}$. Let $(\overline{\mathcal{D}}_i)_i$ be a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{mod}}$ representing $\overline{\mathcal{D}}$, and for each i let $\overline{D}_i \in \widehat{\mathrm{Div}}(X_i)_{\mathbb{R}}$ as in (8.3).

The *essential minimum* of $\overline{\mathcal{D}}$ is defined in the expected way as the quantity

$$\mu^{\mathrm{ess}}(\overline{\mathcal{D}}) = \sup_{V \subsetneq U} \inf_{x \in (U \setminus V)(\overline{K})} h_{\overline{\mathcal{D}}}(x),$$

the supremum being over all the proper closed subsets $V \subsetneq U$. On the other hand, there is no direct extension of the absolute minimum to the quasi-projective setting because the height function of $\overline{\mathcal{D}}$ is only defined on $U(\overline{K})$. For this notion we restrict to the case when $\overline{\mathcal{D}}$ is semipositive and define its *absolute minimum* as the quantity

$$\mu^{\mathrm{abs}}(\overline{\mathcal{D}}) = \sup\{\lambda \in \mathbb{R} \mid \overline{\mathcal{D}} - \lambda[\infty] \text{ is nef}\},$$

in agreement with (2.6).

We need some auxiliary results.

Lemma 8.6. *If $\overline{\mathcal{D}}$ is semipositive and $\mu^{\mathrm{abs}}(\overline{\mathcal{D}}) > -\infty$, then $\overline{\mathcal{D}} - \mu^{\mathrm{abs}}(\overline{\mathcal{D}})[\infty]$ is nef.*

Proof. Let $(\lambda_n)_n$ be a sequence of real numbers converging to $\mu^{\mathrm{abs}}(\overline{\mathcal{D}})$ from below. Then for every n there exists a Cauchy sequence $(\overline{\mathcal{D}}_{n,i})_i$ in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{mod}}$ representing $\overline{\mathcal{D}} - \lambda_n[\infty]$ with $\overline{\mathcal{D}}_{n,i}$ nef for every i. Since $\overline{\mathcal{B}}$ is strictly effective we have that $\|[\infty]\|_{\overline{\mathcal{B}}}$ is a real number. Then for any $\varepsilon > 0$ and every n and i sufficiently large we have

$$\|\overline{\mathcal{D}}_{n,i} - (\overline{\mathcal{D}} - \mu^{\mathrm{abs}}(\overline{\mathcal{D}})\,[\infty])\|_{\overline{\mathcal{B}}} \le \|\overline{\mathcal{D}}_{n,i} - (\overline{\mathcal{D}} - \lambda_n[\infty])\|_{\overline{\mathcal{B}}} + (\mu^{\mathrm{abs}}(\overline{\mathcal{D}}) - \lambda_n)\,\|[\infty]\|_{\overline{\mathcal{B}}} < \varepsilon.$$

It follows that $\overline{\mathcal{D}} - \mu^{\mathrm{abs}}(\overline{\mathcal{D}})[\infty]$ can be represented by a Cauchy sequence of nef model arithmetic $\mathbb{R}$-divisors, and so it is nef. □

Lemma 8.7. *We have $\mu^{\mathrm{ess}}(\overline{\mathcal{D}}) = \lim_{i \to \infty} \mu^{\mathrm{ess}}(\overline{D}_i)$.*

Proof. For any $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$\mu^{\mathrm{ess}}(k\,\overline{\mathcal{D}} + t\,[\infty]) = k\,\mu^{\mathrm{ess}}(\overline{\mathcal{D}}) + t \quad \text{and} \quad \mu^{\mathrm{ess}}(k\overline{D}_i + t\,[\infty]) = k\,\mu^{\mathrm{ess}}(\overline{D}_i) + t, \quad i \in \mathbb{N},$$

and so we can replace without loss of generality $\overline{\mathcal{D}}$ by $k\,\overline{\mathcal{D}}+t\,[\infty]$ and $\overline{\mathcal{D}}_i$ by $k\,\overline{\mathcal{D}}_i+t\,[\infty]$. Since D is big, taking k and t sufficiently large and applying Lemma 2.7 we can assume that $\overline{\mathcal{B}} \le \overline{\mathcal{D}}_i$, first for $i = 1$ and then for every $i \in \mathbb{N}$ using the assumption (8.2).

Since $(\overline{\mathcal{D}}_i)_i$ is Cauchy, there exists a sequence of positive real numbers $(\varepsilon_i)_i$ converging to zero such that

$$\overline{\mathcal{D}}_i - \varepsilon_i\overline{\mathcal{B}} \le \overline{\mathcal{D}}_j \le \overline{\mathcal{D}}_i + \varepsilon_i\overline{\mathcal{B}} \quad \text{for every } 0 \le i \le j. \tag{8.5}$$

Since the support of $\pm(D_j - D_i) + \varepsilon_i B$ does not intersect U, we get from (8.5)

$$h_{\overline{\mathcal{D}}_i}(x) - \varepsilon_i h_{\overline{\mathcal{B}}}(x) \le h_{\overline{\mathcal{D}}_j}(x) \le h_{\overline{\mathcal{D}}_i}(x) + \varepsilon_i h_{\overline{\mathcal{B}}}(x) \quad \text{for every } x \in U(\overline{K}). \tag{8.6}$$

Since $\overline{\mathcal{B}} \le \overline{\mathcal{D}}_i$, there exists a dense open subset $U_i \subset U$ such that $h_{\overline{\mathcal{B}}}(x) \le h_{\overline{\mathcal{D}}_i}(x)$ for every $x \in U_i(\overline{K})$. Using this and taking the limit for $j \to \infty$ in (8.6) we deduce

$$(1 - \varepsilon_i)h_{\overline{\mathcal{D}}_i}(x) \le h_{\overline{\mathcal{D}}}(x) \le (1 + \varepsilon_i)h_{\overline{\mathcal{D}}_i}(x) \quad \text{for every } x \in U_i(\overline{K}).$$

Since U_i is dense, this implies that $(1-\varepsilon_i)\mu^{\mathrm{ess}}(\overline{D}_i) \le \mu^{\mathrm{ess}}(\overline{\mathcal{D}}) \le (1+\varepsilon_i)\mu^{\mathrm{ess}}(\overline{D}_i)$, and the lemma follows by letting $i \to \infty$. $\square$

Lemma 8.8. *Let $\overline{\mathcal{E}} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ such that $E \in \mathrm{Div}(U)^{\mathrm{adel}}_{\mathbb{R}}$ is big and $\overline{\mathcal{D}} - \overline{\mathcal{E}}$ is pseudo-effective. Then $\mu^{\mathrm{ess}}(\overline{\mathcal{D}}) \ge \mu^{\mathrm{ess}}(\overline{\mathcal{E}})$.*

Proof. Let $(\overline{\mathcal{E}}_i)_i$ and $(\overline{\mathcal{M}}_i)_i$ be Cauchy sequences in $\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{mod}}_{\mathbb{R}}$ representing respectively $\overline{\mathcal{E}}$ and $\overline{\mathcal{D}} - \overline{\mathcal{E}}$, with $\overline{\mathcal{M}}_i$ pseudo-effective for every i. Then there is a sequence $(\varepsilon_i)_i$ of real numbers converging to zero such that

$$\overline{\mathcal{M}}_i \le \overline{\mathcal{D}}_i - \overline{\mathcal{E}}_i + \varepsilon_i \overline{\mathcal{B}} \quad \text{for every } i. \tag{8.7}$$

Then $(\overline{\mathcal{D}}_i + \varepsilon_i\overline{\mathcal{B}})_i$ is a Cauchy sequence representing $\overline{\mathcal{D}}$ which by (8.7) satisfies that $(\overline{\mathcal{D}}_i + \varepsilon_i\overline{\mathcal{B}}) - \overline{\mathcal{E}}_i$ is pseudo-effective for every i. By Lemmas 2.16(4) and 8.7 we have

$$\mu^{\mathrm{ess}}(\overline{\mathcal{D}}) = \lim_{i\to\infty} \mu^{\mathrm{ess}}(\overline{D}_i + \varepsilon_i \overline{B}) \ge \lim_{i\to\infty} \mu^{\mathrm{ess}}(\overline{E}_i) = \mu^{\mathrm{ess}}(\overline{\mathcal{E}}).$$

$\square$

The next result is the quasi-projective version of Corollary 5.3, and will be the key ingredient in the proof of our quasi-projective equidistribution theorem.

Proposition 8.9. *Let $\overline{\mathcal{P}}, \overline{\mathcal{E}} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ with $\overline{\mathcal{P}}$ nef and $P \in \mathrm{Div}(U)^{\mathrm{adel}}_{\mathbb{R}}$ big. Assume that there exists a nef $\overline{\mathcal{A}} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ such that $\overline{\mathcal{A}} \pm \overline{\mathcal{E}}$ are nef and $A \in \mathrm{Div}(U)^{\mathrm{adel}}_{\mathbb{R}}$ is big. There exists a constant c_d depending only on d such that*

$$\mu^{\mathrm{ess}}(\overline{\mathcal{P}} + \lambda\overline{\mathcal{E}}) \ge \frac{(\overline{\mathcal{P}}^{d+1})}{(d+1)\operatorname{vol}(P+\lambda E)} + \frac{(\overline{\mathcal{P}}^{d} \cdot \overline{\mathcal{E}})}{(P^d)}\lambda - c_d \frac{(\overline{\mathcal{P}}^{d} \cdot \overline{\mathcal{A}})}{(P^d)} \frac{\lambda^2}{r(P;A)}$$

for every $0 \le \lambda < r(P;A)/2$. In particular, if $E = 0 \in \mathrm{Div}(U)^{\mathrm{adel}}_{\mathbb{R}}$ then

$$\mu^{\mathrm{ess}}(\overline{\mathcal{P}} + \lambda\overline{\mathcal{E}}) \ge \frac{(\overline{\mathcal{P}}^{d+1})}{(d+1)(P^d)} + \frac{(\overline{\mathcal{P}}^{d} \cdot \overline{\mathcal{E}})}{(P^d)}\lambda - c_d \frac{(\overline{\mathcal{P}}^{d} \cdot \overline{\mathcal{A}})}{(P^d)} \frac{\lambda^2}{r(P;A)}.$$

Proof. Let $(\overline{\mathcal{P}}_i)_i, (\overline{\mathcal{M}}_i)_i$ and $(\overline{\mathcal{N}}_i)_i$ be Cauchy sequences in $\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{mod}}_{\mathbb{R}}$ representing respectively $\overline{\mathcal{P}}, \overline{\mathcal{A}} + \overline{\mathcal{E}}$ and $\overline{\mathcal{A}} - \overline{\mathcal{E}}$ and such that $\overline{\mathcal{P}}_i, \overline{\mathcal{M}}_i$ and $\overline{\mathcal{N}}_i$ are nef and defined on the same projective arithmetic variety $\mathcal{X}_i$ for every i. Set

$$\overline{\mathcal{A}}_i = \frac{\overline{\mathcal{M}}_i + \overline{\mathcal{N}}_i}{2} \quad \text{and} \quad \overline{\mathcal{E}}_i = \frac{\overline{\mathcal{M}}_i - \overline{\mathcal{N}}_i}{2}.$$

Then $(\overline{\mathcal{A}}_i)_i$ and $(\overline{\mathcal{E}}_i)_i$ are Cauchy sequences representing $\overline{\mathcal{A}}$ and $\overline{\mathcal{E}}$, and we have that $\overline{A}_i$ and $\overline{A}_i \pm \overline{E}_i$ are nef for every i.

Let $\lambda \in [0, r(P;A)/2)$. By Lemma 8.2 we have $\lambda \in [0, r(P_i;A_i)/2)$ for every sufficiently large i. In particular

$$P_i + \lambda E_i = P_i - \lambda A_i + \lambda(A_i + E_i)$$

is big because $P_i - \lambda A_i$ is big and $A_i + E_i$ is nef, hence pseudo-effective. Then by Corollary 5.3 there exists a constant c_d depending only on d such that

$$\mu^{\mathrm{ess}}(\overline{P}_i + \lambda\overline{E}_i) \ge \frac{(\overline{P}_i^{d+1})}{(d+1)\operatorname{vol}(P_i+\lambda E_i)} + \frac{(\overline{P}_i^{d} \cdot \overline{E}_i)}{(P_i^d)}\lambda - c_d \frac{(\overline{P}_i^{d} \cdot \overline{A}_i)}{(P_i^d)} \frac{\lambda^2}{r(P_i;A_i)}.$$

We conclude by letting $i \to \infty$. $\square$

8.3. **Equidistribution on quasi-projective varieties.** Let $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{adel}}$ with big $D \in \mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{adel}}$, as in the previous section.

Definition 8.10. A *semipositive approximation* of $\overline{\mathcal{D}}$ is a pair $(\phi, \overline{\mathcal{Q}})$ where

(1) $\phi\colon \mathcal{U}' \to \mathcal{U}$ is a birational morphism of normal quasi-projective arithmetic varieties,
(2) $\overline{\mathcal{Q}}$ is a semipositive adelic $\mathbb{R}$-divisor on $\mathcal{U}'$ with big $Q \in \mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{adel}}$,
(3) $\phi^*\overline{\mathcal{D}} - \overline{\mathcal{Q}}$ is pseudo-effective.

A sequence $(x_\ell)_\ell$ in $U(\overline{K})$ is called *generic* if for every closed subset $V \subsetneq U$ there exists $\ell_0 \in \mathbb{N}$ such that $x_\ell \notin V(\overline{K})$ for every $\ell \geq \ell_0$. A generic sequence $(x_\ell)_\ell$ is called $\overline{\mathcal{D}}$*-small* if

$$\lim_{\ell \to \infty} h_{\overline{\mathcal{D}}}(x_\ell) = \mu^{\mathrm{ess}}(\overline{\mathcal{D}}).$$

For every $x \in U(\overline{K})$ the Galois orbit $O(x) \subset X(\overline{K})$ lies in $U(\overline{K})$. Thus for every $v \in \mathfrak{M}_K$ the v-adic Galois orbit $O(x)_v$ lies in U_v^{an}, and in particular $\delta_{O(x)_v}$ is a probability measure on U_v^{an}.

The following is the quasi-projective version of Theorem 5.8.

Theorem 8.11. *Assume that there exists a sequence $(\phi_n\colon \mathcal{U}_n \to \mathcal{U}, \overline{\mathcal{Q}}_n)_n$ of semipositive approximations of $\overline{\mathcal{D}}$ such that*

$$\lim_{n\to\infty} \frac{1}{r(Q_n;\phi_n^*D)}\Big(\mu^{\mathrm{ess}}(\overline{\mathcal{D}}) - \frac{(\overline{\mathcal{Q}}_n^{d+1})}{(d+1)\,(Q_n^d)}\Big) = 0,\ \sup_{n\in\mathbb{N}} \frac{\mu^{\mathrm{ess}}(\overline{\mathcal{D}}) - \mu^{\mathrm{abs}}(\overline{\mathcal{Q}}_n)}{r(Q_n;\phi_n^*D)} < \infty. \quad (8.8)$$

Let $v \in \mathfrak{M}_K$, and for each $n \geq 1$ let $\nu_{n,v}$ be the pushforward to U_v^{an} of the normalized v-adic Monge-Ampère measure $c_1(\overline{\mathcal{Q}}_{n,v})^{\wedge d}/(Q_n^d)$ on $U_{n,v}^{\mathrm{an}}$. Then

(1) the sequence $(\nu_{n,v})_n$ converges weakly to a probability measure $\nu_{\overline{\mathcal{D}},v}$ on U_v^{an},
(2) for every $\overline{\mathcal{D}}$-small generic sequence $(x_\ell)_\ell$ in $U(\overline{K})$, the sequence of probability measures $(\delta_{O(x_\ell)_v})_\ell$ on U_v^{an} converges weakly to $\nu_{\overline{\mathcal{D}},v}$.

Proof. Let $v \in \mathfrak{M}_K$ and $\varphi\colon U_v^{\mathrm{an}} \to \mathbb{R}$ a continuous function with compact support, and let $(x_\ell)_\ell$ be a $\overline{\mathcal{D}}$-small generic sequence in $U(\overline{K})$. We need to show that

$$\lim_{\ell\to\infty} \int_{U_v^{\mathrm{an}}} \varphi\, d\delta_{O(x_\ell)_v} = \lim_{n\to\infty} \int_{U_v^{\mathrm{an}}} \varphi\, d\nu_{n,v}. \quad (8.9)$$

Let $\varepsilon > 0$. Since φ has compact support, we can view it as an element of $C(X_v^{\mathrm{an}})$. By [GM22, Proposition 2.11 and Theorem 2.13], after possibly extending the base field K, there exists a $\mathrm{Gal}(\overline{K}_v/K_v)$-invariant $\varphi_\varepsilon \in C(X_v^{\mathrm{an}})$ such that $|\varphi_\varepsilon - \varphi| < \varepsilon$ on X_v^{an} and such that the adelic divisor $\overline{E} := \overline{0}^{\varphi_\varepsilon}$ as in (2.8) is DSP. Then by Lemma 2.14 there exists $\overline{A} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ such that both $\overline{A}$ and $\overline{A} \pm \overline{E}$ are nef and $A \in \mathrm{Div}(X)_{\mathbb{R}}$ is big. Shrinking $\mathcal{U}$ if necessary, we view these adelic $\mathbb{R}$-divisors on X as elements of $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{int}}$, in which case we respectively denote them by $\overline{\mathcal{E}}$ and $\overline{\mathcal{A}}$.

For each $n \in \mathbb{N}$ set $\widetilde{\mathcal{Q}}_n = \overline{\mathcal{Q}}_n - \mu^{\mathrm{abs}}(\overline{\mathcal{Q}}_n)\,[\infty]$. By Lemma 8.6 we have that $\widetilde{\mathcal{Q}}_n$ is nef. Moreover, the second condition in (8.8) implies

$$\kappa := \sup_{n\in\mathbb{N}} \frac{(\widetilde{\mathcal{Q}}_n^d \cdot \phi_n^*\overline{\mathcal{A}})}{(Q_n^d)} < \infty.$$

We omit the proof, as it is identical to that for Lemma 5.6. In this respect, note that Zhang's inequality remains valid in the quasi-projective setting: this is [YZ26, Theorem 5.3.3], and alternatively it follows from Proposition 8.9 applied with $\lambda = 0$.

Let $n \in \mathbb{N}$ and $\lambda \in (0, r(Q_n;\phi_n^* A)/2)$. By Proposition 8.9 applied with $\overline{\mathcal{P}} = \widetilde{\mathcal{Q}}_n$, there exists a constant c_d depending only on d such that

$$\mu^{\mathrm{ess}}(\widetilde{\mathcal{Q}}_n + \lambda\, \phi_n^*\overline{\mathcal{E}}) \ge \frac{(\widetilde{\mathcal{Q}}_n^{d+1})}{(d+1)\,(Q_n^d)} + \frac{(\widetilde{\mathcal{Q}}_n^d \cdot \phi_n^*\overline{\mathcal{E}})}{(Q_n^d)}\, \lambda - c_d\, \kappa\, \frac{\lambda^2}{r(Q_n;\phi_n^* A)}.$$

Since $(\widetilde{\mathcal{Q}}_n^{d+1}) = (\overline{\mathcal{Q}}_n^{d+1}) - (d+1)\, \mu^{\mathrm{abs}}(\overline{\mathcal{Q}}_n)\, (Q_n^d)$ and $\mu^{\mathrm{ess}}(\widetilde{\mathcal{Q}}_n + \lambda\, \phi_n^*\overline{\mathcal{E}}) = \mu^{\mathrm{ess}}(\overline{\mathcal{Q}}_n + \lambda\, \phi_n^*\overline{\mathcal{E}}) - \mu^{\mathrm{abs}}(\overline{\mathcal{Q}}_n)$ we obtain

$$\mu^{\mathrm{ess}}(\overline{\mathcal{Q}}_n + \lambda\, \phi_n^*\overline{\mathcal{E}}) \ge \frac{(\overline{\mathcal{Q}}_n^{d+1})}{(d+1)\,(Q_n^d)} + \frac{(\widetilde{\mathcal{Q}}_n^d \cdot \phi_n^*\overline{\mathcal{E}})}{(Q_n^d)} \lambda - c_d\, \kappa\, \frac{\lambda^2}{r(Q_n;\phi_n^* A)}.$$

On the other hand we have

$$\mu^{\mathrm{ess}}(\overline{\mathcal{D}}) + \lambda \liminf_{\ell\to\infty} h_{\overline{\mathcal{E}}}(x_\ell) = \liminf_{\ell\to\infty} h_{\overline{\mathcal{D}}+\lambda\overline{\mathcal{E}}}(x_\ell) \ge \mu^{\mathrm{ess}}(\overline{\mathcal{D}} + \lambda\overline{\mathcal{E}}) \ge \mu^{\mathrm{ess}}(\overline{\mathcal{Q}}_n + \lambda\, \phi_n^*\overline{\mathcal{E}}),$$

where the last inequality is given by Lemma 8.8. Therefore

$$\liminf_{\ell\to\infty} h_{\overline{\mathcal{E}}}(x_\ell) \ge \Big(\frac{(\overline{\mathcal{Q}}_n^{d+1})}{(d+1)(Q_n^d)} - \mu^{\mathrm{ess}}(\overline{\mathcal{D}}) \Big) \frac{1}{\lambda} + \frac{(\widetilde{\mathcal{Q}}_n^d \cdot \phi_n^*\overline{\mathcal{E}})}{(Q_n^d)} - c_d\, \kappa\, \frac{\lambda}{r(Q_n;\phi_n^* A)}. \quad (8.10)$$

It follows from the first condition in (8.8) and Lemma 1.2 that

$$\lim_{n\to\infty} \frac{1}{r(Q_n;\phi_n^* A)} \Big(\mu^{\mathrm{ess}}(\overline{\mathcal{D}}) - \frac{(\overline{\mathcal{Q}}_n^{d+1})}{(d+1)(Q_n^d)} \Big) = 0.$$

Therefore, applying the inequality (8.10) to a suitable choice of $\lambda = \lambda_n$ and taking the supremum limit for $n \to \infty$ gives

$$\liminf_{\ell\to\infty} h_{\overline{\mathcal{E}}}(x_\ell) \ge \limsup_{n\to\infty} \frac{(\widetilde{\mathcal{Q}}_n^d \cdot \phi_n^*\overline{\mathcal{E}})}{(Q_n^d)}. \quad (8.11)$$

To conclude we adapt the arguments in the proof of [YZ26, Theorem 5.4.3]. Let $n \in \mathbb{N}$ and choose a Cauchy sequence $(\overline{\mathcal{Q}}_{n,i})_i$ in $\widehat{\mathrm{Div}}(\mathcal{U}_n)^{\mathrm{mod}}$ representing $\widetilde{\mathcal{Q}}_n$ and such that $\overline{\mathcal{Q}}_{n,i}$ is nef for every i. Let $\mathcal{X}_{n,i}$ be a projective arithmetic variety on which $\overline{\mathcal{Q}}_{n,i}$ is defined, denote by $X_{n,i}$ its generic fiber, and set $\overline{Q}_{n,i} \in \widehat{\mathrm{Div}}(X_{n,i})_{\mathbb{R}}$ as in (8.3).

Let $v \in \mathfrak{M}_K$. By (8.4), the v-adic Monge-Ampère measures $c_1(\overline{Q}_{n,i,v})^{\wedge d}$ converge to that of $\widetilde{\mathcal{Q}}_n$, which coincides with that of $\overline{\mathcal{Q}}_n$. Then

$$\begin{aligned}
\frac{(\widetilde{\mathcal{Q}}_n^d \cdot \phi_n^*\overline{\mathcal{E}})}{(Q_n^d)} &= \frac{n_v}{(Q_n^d)} \lim_{i\to\infty} \int_{X_{n,i,v}^{\mathrm{an}}} \varphi_\varepsilon\, c_1(\overline{Q}_{n,i,v})^{\wedge d} \\
&\ge \frac{n_v}{(Q_n^d)} \lim_{i\to\infty} \int_{X_{n,i,v}^{\mathrm{an}}} \varphi\, c_1(\overline{Q}_{n,i,v})^{\wedge d} - n_v\varepsilon \\
&= \frac{n_v}{(Q_n^d)} \int_{U_{n,v}^{\mathrm{an}}} \varphi\, c_1(\overline{\mathcal{Q}}_{n,v})^{\wedge d} - n_v\varepsilon = n_v \int_{U_v^{\mathrm{an}}} \varphi\, d\nu_{n,v} - n_v\varepsilon,
\end{aligned}$$

where in these integrals we write φ_ε and φ for their pullbacks to $X_{n,i,v}^{\mathrm{an}}$ and $U_{n,v}^{\mathrm{an}}$. Moreover

$$n_v\varepsilon + n_v \int_{U_v^{\mathrm{an}}} \varphi\, d\delta_{O(x_\ell)_v} \ge n_v \int_{X_v^{\mathrm{an}}} \varphi_\varepsilon\, d\delta_{O(x_\ell)_v} = h_{\overline{\mathcal{E}}}(x_\ell) \quad \text{for every } \ell \in \mathbb{N}.$$

Combining this with (8.11) and letting $\varepsilon \to 0$ we obtain

$$\liminf_{\ell\to\infty} \int_{U_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} \geq \limsup_{n\to\infty} \int_{U_{n,v}^{\mathrm{an}}} \varphi \, d\nu_{n,v},$$

and we deduce (8.9) by applying this to $-\varphi$. □

The next consequence is the number field case of the Yuan and Zhang's equidistribution theorem in the quasi-projective setting [YZ26, Theorem 5.4.3].

Corollary 8.12. *Assume that $\overline{\mathcal{D}}$ is nef and that*

$$\mu^{\mathrm{ess}}(\overline{\mathcal{D}}) = \frac{(\overline{\mathcal{D}}^{d+1})}{(d+1)(D^d)}.$$

Then for every $v \in \mathfrak{M}_K$ and every $\overline{\mathcal{D}}$-small generic sequence $(x_\ell)_\ell$ in $U(\overline{K})$ the sequence of probability measures $(\delta_{O(x_\ell)_v})_\ell$ on U_v^{an} converges weakly to $c_1(\overline{\mathcal{D}}_v)^{\wedge d}/(D^d)$.

Proof. Apply Theorem 8.11 to the constant sequence $(\phi_n, \overline{\mathcal{Q}}_n) = (\mathrm{Id}_{\mathcal{U}}, \overline{\mathcal{D}})$, $n \in \mathbb{N}$. □

Remark 8.13. We assume throughout that $\mathcal{U}$ is normal to be able to work with $\mathbb{R}$-divisors. Nevertheless, both Theorem 8.11 and Corollary 8.12 can be applied to an adelic divisor $\overline{\mathcal{D}}$ on an arbitrary quasi-projective arithmetic variety $\mathcal{U}$ over $\mathrm{Spec}(\mathcal{O}_K)$ just shrinking to a normal open subset.

Recently, Biswas proved the differentiability of the arithmetic volume function and deduced a quasi-projective version of Chen's equidistribution theorem [Bis24]. It would be interesting to check if this result also follows from Theorem 8.11 by adapting the arguments we used in the proof of Corollary 4.12.

Appendix A. Auxiliary results on convex analysis

Here we recall the constructions and properties from convex analysis that are used in our study of toric varieties in Section 6. We also establish some auxiliary results, most notably Proposition A.3 concerning the rate of the decay of the sup-level sets of a concave function as the level approaches the maximum value.

Fix an integer $d \geq 1$ and let $C \subset \mathbb{R}^d$ be a *convex body*, that is a compact convex subset with nonempty interior.

Definition A.1. For a linear functional $u \in (\mathbb{R}^d)^\vee$ we denote by $w(C,u)$ the length of the interval $u(C) \subset \mathbb{R}$. The *width* of C is defined as

$$w(C) = \inf_{u \in S^{d-1}} w(C,u),$$

where S^{d-1} denotes the unit sphere of $(\mathbb{R}^d)^\vee \simeq \mathbb{R}^d$.

For another convex body $B \subset \mathbb{R}^d$, the *inradius* of C with respect to B is defined as

$$r(C;B) = \sup\{\lambda \in \mathbb{R}_{>0} \mid \exists\, x \in \mathbb{R}^d \text{ such that } \lambda B + x \subset C\}.$$

When B is the unit ball of $\mathbb{R}^d$, it is the classical inradius from Euclidean geometry.

The inradius and the width can be compared up to scalar factors: there are constants $c_1, c_2 > 0$ depending only on d and B such that

$$c_1\, w(C) \leq r(C;B) \leq c_2\, w(C). \tag{A.1}$$

The first inequality comes from [BF87, page 86, inequality (9)] whereas the second is clear from the definitions.

Let $f\colon C \to \mathbb{R}$ be a concave function and set $\mu = \sup_{x\in C} f(x)$. For each $t \leq \mu$ we denote by

$$S_t(f) = \{x \in C \mid f(x) \geq t\}$$

the corresponding sup-level set. It is a nonempty compact convex subset of C that is a convex body whenever $t < \mu$. We also set $C_{\max} = S_\mu(f)$.

We next introduce the basic objects of the differential analysis of concave functions.

Definition A.2. For $x_0 \in C$, the *sup-differential* of f at x_0 is the closed convex subset of $(\mathbb{R}^d)^\vee$ defined as

$$\partial f(x_0) = \{u \in (\mathbb{R}^d)^\vee \mid \langle u, x - x_0\rangle \geq f(x) - f(x_0) \text{ for every } x \in C\}.$$

Its elements are called the *sup-gradients* of f at x_0.

A point $x_0 \in C$ lies in $C_{\max}$ if and only if $0 \in \partial f(x_0)$. When this is the case, the point 0 is *not* a vertex of $\partial f(x_0)$ if and only if there exists $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$ such that both u and $-u$ belong to this sup-differential or equivalently, if and only if

$$f(x) \leq \mu - |\langle u, x - x_0\rangle| \quad \text{for every } x \in C. \tag{A.2}$$

This condition does not depend on the choice of $x_0 \in C_{\max}$: if (A.2) holds then $\langle u, x_1 - x_0\rangle = 0$ for every $x_1 \in C_{\max}$, and so this inequality also holds with x_0 replaced by x_1.

The next proposition is a rigidity result that allows to determine when a concave function admits a "Canadian tent" upper bound like (A.2) in terms of the rate of decay of the inradius or the width of its sup-level sets as the level approaches its maximum.

Proposition A.3. *The following conditions are equivalent:*

(1) for any convex body $B \subset \mathbb{R}^d$ we have $\displaystyle\lim_{t\to\mu} \frac{\mu - t}{r(S_t(f); B)} = 0$,

(2) $\displaystyle\lim_{t\to\mu} \frac{\mu - t}{w(S_t(f))} = 0$,

(3) for every $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$ we have $\displaystyle\lim_{t\to\mu} \frac{\mu - t}{w(S_t(f), u)} = 0$,

(4) for any $x_0 \in C_{\max}$ we have that $0 \in (\mathbb{R}^d)^\vee$ is a vertex of $\partial f(x_0)$.

Its proof relies on the next two lemmas.

Lemma A.4. *With the previous notation, we have*

(1) for every $u \in (\mathbb{R}^d)^\vee\setminus\{0\}$ the function $t \in (-\infty, \mu) \mapsto \dfrac{\mu - t}{w(S_t(f), u)}$ is non-increasing,

(2) the function $t \in (-\infty, \mu) \mapsto \dfrac{\mu - t}{w(S_t(f))}$ is non-increasing.

Proof. First suppose that $d = 1$. Then choose $x_0 \in C_{\max}$ and for each pair $t, t' \in \mathbb{R}$ with $t' < t < \mu$ consider the affine map $\iota\colon \mathbb{R} \to \mathbb{R}$ defined as

$$\iota(x) = \frac{t - t'}{\mu - t'}\, x_0 + \frac{\mu - t}{\mu - t'}\, x.$$

It follows from the concavity of f that $\iota(S_{t'}(f)) \subset S_t(f)$. Denoting by ℓ the Lebesgue measure on $\mathbb{R}$, this gives

$$\frac{\mu - t}{\mu - t'}\, \ell(S_{t'}(f)) \leq \ell(S_t(f)). \tag{A.3}$$

Now let d be any positive integer. Take $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$ and consider the direct image of f with respect to u, which is the concave function $u_* f \colon u(C) \to \mathbb{R}$ defined as

$$u_*f(y) = \sup\{f(x) \mid x \in C \text{ such that } \langle u, x\rangle = y\} \quad \text{for every } y \in u(C).$$

Clearly $\sup_{y\in u(C)} u_*f(y) = \mu$ and $S_t(u_*f) = u(S_t(f))$ for every $t \leq \mu$. Then for any $t' < t < \mu$ the inequality (A.3) gives

$$\frac{\mu - t'}{w(S_{t'}(f), u)} = \frac{\mu - t'}{\ell(S_{t'}(u_*f))} \geq \frac{\mu - t}{\ell(S_t(u_*f))} = \frac{\mu - t}{w(S_t(f), u)}, \tag{A.4}$$

proving (1). The statement (2) follows by choosing $u \in S^{d-1}$ such that $w(S_t(f), u) = w(S_t(f))$ and applying (A.4) to show

$$\frac{\mu - t'}{w(S_{t'}(f))} \geq \frac{\mu - t'}{w(S_{t'}(f), u)} \geq \frac{\mu - t}{w(S_t(f), u)} = \frac{\mu - t}{w(S_t(f))},$$

as stated. ☐

Lemma A.5. *Let $x_0 \in C_{\max}$ and $t < \mu$. Then for every $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$ we have*

$$f(x) \leq \mu - \frac{\mu - t}{w(S_t(f), u)} \,|\langle u, x - x_0\rangle| \quad \textit{for every } x \in C \setminus S_t(f).$$

Proof. Let $x \in C \setminus S_t(f)$ and set $t' = f(x) < t$. Since both x and x_0 lie in $S_{t'}(f)$ we have $|\langle u, x - x_0\rangle| \leq w(S_{t'}(f), u)$. Combining this with Lemma A.4(1) we get

$$\mu - f(x) \geq \frac{\mu - t'}{w(S_{t'}(f), u)} \,|\langle u, x - x_0\rangle| \geq \frac{\mu - t}{w(S_t(f), u)} \,|\langle u, x - x_0\rangle|,$$

which gives the statement. ☐

Proof of Proposition A.3. The equivalence between (1) and (2) follows from the inequalities (A.1), and clearly (2) implies (3).

Now assume (3). If (4) does not hold, then there exists $x_0 \in C_{\max}$ such that $0 \in \partial f(x_0)$ is not a vertex of this convex subset, and so we can take $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$ such that $\mu - |\langle u, x - x_0\rangle| \geq f(x)$ for every $x \in C$ as in (A.2).

For each $t < \mu$ choose $y \in S_t(u_*f) = u(S_t(f))$ such that

$$|y - \langle u, x_0\rangle| \geq \frac{1}{2}\, w(S_t(f), u).$$

Taking $x \in S_t(f)$ with $\langle u, x\rangle = y$ we have $|\langle u, x - x_0\rangle| \geq w(S_t(f), u)/2$ and so

$$\frac{\mu - t}{w(S_t(f), u)} \geq \frac{\mu - f(x)}{w(S_t(f), u)} \geq \frac{|\langle u, x - x_0\rangle|}{w(S_t(f), u)} \geq \frac{1}{2},$$

which contradicts (3) and thus implies (4).

To close the loop, we show that (4) implies (2). For this suppose that (2) does not hold, which by Lemma A.4 implies that there exists $c > 0$ such that

$$\frac{\mu - t}{w(S_t(f))} \geq c \quad \text{for every } t < \mu.$$

In particular $\dim(C_{\max}) < d$ since otherwise $w(S_t(f)) \geq w(C_{\max}) > 0$ for every $t < \mu$.

Take sequences $(t_k)_k$ in $(-\infty, \mu)$ and $(u_k)_k$ in S^{d-1} with $\lim_{k\to\infty} t_k = \mu$ such that $w(S_{t_k}(f)) = w(S_{t_k}(f), u_k)$ for every k. By the compacity of S^{d-1} we can assume that $\lim_{k\to\infty} u_k = u$ for a point $u \in S^{d-1}$. Take also $x_0 \in C_{\max}$. By Lemma A.5 we have

$$\mu - c\,|\langle u_k, x - x_0\rangle| \geq f(x) \quad \text{for every } x \in C \setminus S_{t_k}(f).$$

Now let $x \in C \setminus C_{\max}$. Then $x \notin S_{t_k}(f)$ for $k \gg 1$ and so

$$\mu - c\,|\langle u, x - x_0\rangle| = \lim_{k\to\infty} \mu - c\,|\langle u_k, x - x_0\rangle| \geq f(x).$$

Since $\dim(C_{\max}) < d$, this inequality extends to $x \in C_{\max}$ by continuity. Therefore 0 is not a vertex of $\partial f(x_0)$ and so (4) does not hold. □

Definition A.6. The concave function f is said to be *wide (at its maximum)* if it verifies any of the equivalent conditions in Proposition A.3.

Now a ssume that the considered concave function decomposes as a finite sum

$$f = \sum_{i\in I} n_i f_i \tag{A.5}$$

where each n_i is a positive real number and $f_i \colon C \to \mathbb{R}$ a concave function.

Definition A.7. A *balanced family of sup-gradients* for the decomposition (A.5) is a family of vectors

$$u_i \in (\mathbb{R}^n)^\vee, \quad i \in I,$$

such that there exists $x_0 \in C_{\max}$ with $u_i \in \partial f_i(x_0)$ for every i and $\sum_{i\in I} n_i u_i = 0$.

Proposition A.8. *The decomposition $f = \sum_{i\in I} n_i f_i$ admits a balanced family of sup-gradients. If f is wide then this family is unique.*

Proof. For the first statement, the decomposition of f implies the decomposition of its sup-differential at a point $x_0 \in C$ as the Minkowski sum

$$\partial f(x_0) = \sum_{i\in I} n_i\, \partial f_i(x_0), \tag{A.6}$$

see for instance [BPS14, Proposition 2.3.9]. If $x_0 \in C_{\max}$ then $0 \in \partial f(x_0)$, and so we obtain a balanced family of sup-gradients by considering any decomposition of this vector according to (A.6). This proves the first statement.

If f is wide then $0 \in \partial f(x_0)$ is a vertex, and so the second statement is given by [BPRS19, Proposition 3.15]. □

We denote by $\mathrm{MV}(C_1, \dots, C_d)$ the mixed volume of a family of d convex bodies of $\mathbb{R}^d$, and by $\mathrm{MI}(f_0, \dots, f_d)$ the mixed integral of a family of concave functions on convex bodies of $\mathbb{R}^d$, both with respect to the Lebesgue measure of $\mathbb{R}^d$. They are respectively defined as alternating sums of Minkowski sums of convex bodies and sup-convolutions of concave functions, see [BPS14, Definitions 2.7.14 and 2.7.16] for precisions.

The next lemma gives the continuity of the mixed integral with respect to the approximation of the domains of the involved concave functions. The proof is straightforward from the behavior of sup-convolutions with respect to restrictions of domains and the continuity of the integral of a concave function on a convex body with respect to the approximation of its domain.

Lemma A.9. *For $i = 0, \dots, d$ let $f_i \colon C_i \to \mathbb{R}$ be a concave function on a convex body, and $(C_{i,n})_n$ a sequence of convex bodies approaching C_i uniformly from inside. Then*

$$\lim_{n\to\infty} \mathrm{MI}(f_0|_{C_{0,n}}, \dots, f_d|_{C_{d,n}}) = \mathrm{MI}(f_0, \dots, f_d).$$

Finally, the next result allows to compute the mixed integral when all but one of the involved concave functions are equal and affine. Recall that the *Legendre-Fenchel dual* of a concave function on a convex body $f\colon C \to \mathbb{R}$ is the concave function $f^\vee\colon (\mathbb{R}^d)^\vee \to \mathbb{R}$ defined as

$$f^\vee(u) = \inf_{x\in C} \langle u, x\rangle - f(x). \tag{A.7}$$

Lemma A.10. *Let $f\colon C \to \mathbb{R}$ an affine function on a convex body with linear part $u \in (\mathbb{R}^d)^\vee$ and constant $c \in \mathbb{R}$. Then for any concave function on a convex body $g\colon B \to \mathbb{R}$ we have*

$$\mathrm{MI}(f,\dots,f,g) = \mathrm{MI}(u|_C,\dots,u|_C,u|_B) + c\,d\,\mathrm{MV}(C,\dots,C,B) - d!\,\mathrm{vol}(C)\,g^\vee(u).$$

Proof. The proof is based on [Gua18, Section 1.3] and we will freely use the notation therein. This requires that both C and B are lattice polytopes, which we now suppose.

By Corollary 1.10 in *loc. cit.* we have

$$\mathrm{MI}(f,\dots,f,g) = \mathrm{MI}(u|_C,\dots,u|_C,g) + c\,d\,\mathrm{MV}(C,\dots,C,B), \tag{A.8}$$

and by Proposition 1.5 in *loc. cit.*, the mixed real Monge-Ampère measure of $u|_C$ is

$$\mathrm{M}\mathcal{M}(u|_C,\dots,u|_C) = d!\,\mathrm{vol}(C)\,\delta_u$$

with δ_u the Dirac measure at the point $u \in (\mathbb{R}^d)^\vee$. Hence applying the recursive formula of Theorem 1.6 in *loc. cit.* to g and to $u|_B$ we get

$$\mathrm{MI}(u|_C,\dots,u|_C,g) - \mathrm{MI}(u|_C,\dots,u|_C,u|_B) = d!\,\mathrm{vol}(C)\,((u|_B)^\vee(u) - g^\vee(u)). \tag{A.9}$$

The statement follows in this case from (A.8) and (A.9) together with the fact that $(u|_B)^\vee(u) = \inf_{x\in B}(\langle u,x\rangle - \langle u,x\rangle) = 0$.

The case when C and B are arbitrary convex bodies is deduced from the previous using the invariance of the formula with respect to homothecies and its continuity with respect to uniform approximations. □

Remark A.11. For any $u \in (\mathbb{R}^d)^\vee$ and convex bodies $C_1, C_2 \subset \mathbb{R}^d$, the sup-convolution of the restrictions $u|_{C_1}$ and $u|_{C_2}$ coincides with the restriction $u|_{C_1+C_2}$, that is

$$u|_{C_1} \boxplus u|_{C_2} = u|_{C_1+C_2}.$$

Hence the mixed integral $\mathrm{MI}(u|_C,\dots,u|_C,u|_B)$ can be written as an alternating sum of integrals of the linear function u on the Minkowski sum of several copies C and B. In particular, the map $u \mapsto \mathrm{MI}(u|_C,\dots,u|_C,u|_B)$ is linear.

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